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SOME REMARKS ON THE BIENERGY OF PULL-BACK VECTOR FIELDS

Abstract. The problem studied in this paper is related to the bienergy of a pull-back vector field from a Riemannian manifold (M, g) to its tangent bundle TN equipped with the Sasaki metric h^s. We show that a pull-back vector field on a compact manifold (M, g) is biharmonic if and only if it is harmonic. We also investigate the bienergy of a pull-back vector field, as a map from (M, g) to (TN, h^s).

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1 Introduction

In vector calculus and vector physics, a vector field is the assignment of a vector to each point in a given space. As an illustration, the position vector of a space curve is specified only for a smaller subset of the ambient space. A vector field is a special case of a vector-valued function, whose domain's dimension has no relationship to the size of its range. Think about the movement through a spatial region. Every point has a specific velocity connected with it at any given time, hence every flow has a vector field associated with it. Numerous phenomenological formulations and applications in this direction have been researched in recent decades (see [1, 12–15, 17–21, 23, 24]). In a broader sense, vector fields are defined on differentiable manifolds, which are spaces that, at greater sizes, may have a more intricate structure than Euclidean space. In this configuration, every point on the manifold (i.e., a segment of the tangent bundle to the manifold) has a tangent vector provided by a vector field. One type of tensor field is a vector field. Let $\varphi : M \to N$ be a smooth map between the smooth manifolds M, N. The map φ induces the pull-back vector field $V : M \to TN$ in the case where M, N are Riemannian manifolds and TN is the tangent bundle equipped with the Sasaki metric. The motivation of this paper is to study the harmonicity and biharmonicity of the pull-back vector field $V : (M, g) \to (TN, h^s)$. The energy functional of the map φ between Riemannian manifolds has been widely investigated by several researchers (see [2–11]). Biharmonic maps are inherently harmonic maps. Proper biharmonic mappings are defined as non-harmonic biharmonic maps. The idea of biharmonic maps has garnered increasing attention over the past ten years and falls under two primary categories for further investigation.

In this paper, we deal with these problems. We show that if (M, g) is a compact oriented m-dimensional Riemannian manifold and the map φ is harmonic, then the pull-back vector field $V \in \Gamma(\varphi^{-1}TN)$ is harmonic if and only if V is parallel.

In the biharmonicity, we show that if (M, g) is a compact oriented m-dimensional Riemannian manifold and the map φ is harmonic, then the pull-back vector field $V \in \Gamma(\varphi^{-1}TN)$ is biharmonic if and only if V is harmonic.

1.1 Harmonic maps

Consider a smooth map $\phi : (M^m, g) \to (N^n, h)$ between two Riemannian manifolds, then the energy functional is defined by

$$E(\phi) = \frac{1}{2}\int_M |d\phi|^2 v_g$$

(or over any compact subset $K \subset M$).

A map is called harmonic if it is a critical point of the energy functional E (or $E(K)$ for all compact subsets $K \subset M$). For any smooth variation $\{\phi\}_{t\in I}$ of ϕ with $\phi_0 = \phi$ and $V = \frac{d\phi_t}{dt}\big|_{t=0}$, we have

$$\frac{d}{dt}\,E(\phi_t)\big|_{t=0} = -\int_M h(\tau(\phi), V)v_g,$$

where $\tau(\phi) = tr_g\nabla d\phi$ is the tension field of ϕ. Therefore, the following theorem is valid.

Theorem 1.1. *A smooth map $\phi : (M^m, g) \to (N^n, h)$ is harmonic if and only if*

$$\tau(\phi) = 0. \tag{1.1}$$

If $(x^i)_{1\le i\le m}$ and $(y^\alpha)_{1\le\alpha\le n}$ denote local coordinates on M and N, respectively, then equation (1.1) takes the form

$$\tau(\phi)^\alpha = \left(\Delta\phi^\alpha + g^{ij}\overset{N}{\Gamma}{}^\alpha_{\beta\gamma}\frac{\partial\phi^\beta}{\partial x^i}\frac{\partial\phi^\gamma}{\partial x^j}\right) = 0,$$

where

$$\Delta\phi^\alpha = \frac{1}{\sqrt{|g|}}\frac{\partial}{\partial x^i}\left(\sqrt{|g|}\,g^{ij}\frac{\partial\phi^\alpha}{\partial x^j}\right)$$

is the Laplace operator on (M^m, g) and $\overset{N}{\Gamma}{}^\alpha_{\beta\gamma}$ are the Christoffel symbols on N.

1.2 Biharmonic maps

Definition 1.1. A map $\phi : (M^m, g) \to (N^n, h)$ between Riemannian manifolds is called biharmonic if it is a critical point of the bienergy functional

$$E_2(\phi) = \frac{1}{2}\int_M |\tau(\phi)|^2 v_g.$$

We have

$$\frac{d}{dt} E_2(\phi_t)\big|_{t=0} = -\int_M h(\tau_2(\phi), V) v_g.$$

The Euler–Lagrange equation attached to bienergy is given by the vanishing bitension field

$$\tau_2(\phi) = -J_\phi(\tau(\phi)) = -\big(\Delta^\phi \tau(\phi) + tr_g R^N(\tau(\phi), d\phi)d\phi\big),$$

where J_ϕ is the Jacobi operator defined by

$$J_\phi : \Gamma(\phi^{-1}(TN)) \to \Gamma(\phi^{-1}(TN)),$$
$$V \mapsto \Delta^\phi V + tr_g R^N(V, d\phi)d\phi.$$

The biharmonic map, introduced by Eelles and Sampson in 1964, is a generalization of harmonic maps. For background on harmonic and biharmonic maps, we refer to [6, 16, 22].

2 Basic notions and definition on TM

Let (M, g) be an n-dimensional Riemannian manifold and (TM, π, M) be its tangent bundle. A local chart $(U, x^i)_{i=1,\dots,n}$ on M induces a local chart $(\pi^{-1}(U), x^i, y^i)_{i=1,\dots,n}$ on TM. Denote by Γ_{ij}^k the Christoffel symbols of g and by ∇ the Levi–Civita connection of g.

We have two complementary distributions on TM, the vertical distribution $\mathcal{V}$ and the horizontal distribution $\mathcal{H}$ defined by

$$\mathcal{V}_{(x,u)} = \ker(d\pi_{(x,u)}) = \Big\{a^i \frac{\partial}{\partial y^i}_{|(x,u)};\ a^i \in \mathbb{R}\Big\},$$
$$\mathcal{H}_{(x,u)} = \Big\{\frac{\partial}{\partial x^i}_{|(x,u)} - a^i u^j \Gamma_{ij}^k \frac{\partial}{\partial y^k}_{|(x,u)};\ a^i \in \mathbb{R}\Big\},$$

where $(x, u) \in TM$ such that

$$T_{(x,u)}TM = \mathcal{H}_{(x,u)} \oplus \mathcal{V}_{(x,u)}.$$

Let $X = X^i \frac{\partial}{\partial x^i}$ be a local vector field on M. The vertical and the horizontal lifts of X are defined by

$$X^V = X^i \frac{\partial}{\partial y^i},$$
$$X^H = X^i \frac{\delta}{\delta x^i} = X^i \Big\{\frac{\partial}{\partial x^i} - y^j \Gamma_{ij}^k \frac{\partial}{\partial y^k}\Big\}.$$

For consequences, we have $(\frac{\partial}{\partial x^i})^H = \frac{\delta}{\delta x^i}$ and $(\frac{\partial}{\partial x^i})^V = \frac{\partial}{\partial y^i}$, then $(\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^i})_{i=1,\dots,n}$ is a local adapted frame in TTM.

Definition 2.1. The Sasaki metric g^s on the tangent bundle TM of M is given by

1. $g^s(X^H, Y^H) = g(X, Y) \circ \pi$,
2. $g^s(X^H, Y^V) = 0$,
3. $g^s(X^V, Y^V) = g(X, Y) \circ \pi$

for all vector fields $X, Y \in \Gamma(TM)$.

Proposition 2.1 ([11]). *Let (M, g) be a Riemannian manifold and $\widehat{\nabla}$ be the Levi–Civita connection of the tangent bundle (TM, g^s) equipped with the Sasaki metric. Then*

$$\begin{aligned}
(\widehat{\nabla}_{X^H} Y^H)_{(x,u)} &= (\nabla_X Y)^H_{(x,u)} - \frac{1}{2}\left(R_x(X,Y)u\right)^V, \\
(\widehat{\nabla}_{X^H} Y^V)_{(x,u)} &= (\nabla_X Y)^V_{(x,u)} + \frac{1}{2}\left(R_x(u,Y)X\right)^H, \\
(\widehat{\nabla}_{X^V} Y^H)_{(x,u)} &= \frac{1}{2}\left(R_x(u,X)Y\right)^H, \\
(\widehat{\nabla}_{X^V} Y^V)_{(x,u)} &= 0
\end{aligned}$$

for all vector fields $X, Y \in \Gamma(TM)$ and $(x,u) \in TM$.

3 Harmonicity of pull-back vector fields

Lemma 3.1 ([4]). *Let (M, g) be a Riemannian manifold. If $X, Y \in \Gamma(TM)$ are the vector fields and $(x,u) \in TM$ such that $X_x = u$, then we have*

$$d_x X(Y_x) = Y^H_{(x,u)} + (\nabla_Y X)^V_{(x,u)}.$$

Lemma 3.2. *Let $\varphi : (M^m, g) \to (N^n, h)$ be a smooth map between the Riemannian manifolds. The map φ induces the pull-back vector fields*

$$\begin{aligned}
V : (M, g) &\to (TN, h^s), \\
x &\to (\varphi(x), Y_{\varphi(x)})
\end{aligned}$$

for all vector field $V \in \Gamma(\varphi^{-1}TN)$ and $X \in \Gamma(TM)$, and we have

$$dV(X) = (d\varphi(X))^H + (\nabla^\varphi_X V)^V.$$

Proof. From Lemma 3.1, we have

$$\begin{aligned}
dV(X_x) &= d(Y \circ \varphi)(X_x) = dY(d\varphi(X_x)) \\
&= (d\varphi(X))^H_{(x,u)} + \left(\nabla_{d\varphi(X)} Y \circ \varphi\right)^V_{(x,u)} \\
&= (d\varphi(X))^H_{(x,u)} + (\nabla^\varphi_X V)^V_{(x,u)}.
\end{aligned}$$

□

Proposition 3.1. *The tension field of the pull-back vector fields $V \in \Gamma(\varphi^{-1}TN)$ is given by*

$$\tau(V) = \Big(\tau(\varphi) + tr_g R^N(V, \nabla^\varphi_* V) d\varphi(*)\Big)^H + \left(tr_g(\overset{\varphi}{\nabla}{}^2 V)\right)^V.$$

Proof. Let $x \in M$ and $\{e_i\}_{i=1}^n$ be a local orthonormal frame on M such that $\nabla_{e_i} e_j = 0$ at x and $X_x = u$. By summing over i, we have

$$\begin{aligned}
\tau(V) &= \left\{\nabla^V_{e_i} dV(e_i)\right\} \\
&= \Big\{\nabla^{TN}_{(d\varphi(e_i))^H}(d\varphi(e_i))^H + \nabla^{TN}_{(d\varphi(e_i))^H}(\nabla^\varphi_{e_i} V)^V + \nabla^{TN}_{(\nabla^\varphi_{e_i} V)^V}(\nabla^\varphi_{e_i} V)^V + \nabla^{TN}_{(\nabla^\varphi_{e_i} V)^V}(d\varphi(e_i))^H\Big\} \\
&= \Big\{\left(\nabla_{d\varphi(e_i)} d\varphi(e_i)\right)^H - \frac{1}{2}\left(R(d\varphi(e_i), d\varphi(e_i))v\right)^V + \left(\nabla_{d\varphi(e_i)}(\nabla^\varphi_{e_i} V)\right)^V \\
&\qquad + \frac{1}{2}\left(R_x(v, \nabla^\varphi_{e_i} V) d\varphi(e_i)\right)^H + \frac{1}{2}\left(R(v, \nabla^\varphi_{e_i} V) d\varphi(e_i)\right)^H\Big\},
\end{aligned}$$

and then

$$\tau(V) = \Big(\tau(\varphi) + tr_g R^N(V, \nabla^\varphi_* V) d\varphi(*)\Big)^H + \left(tr_g(\overset{\varphi}{\nabla}{}^2 V)\right)^V.$$

□

Theorem 3.1. *The pull-back vector field $V \in \Gamma(\varphi^{-1}TN)$ is harmonic if and only if*

$$\tau(\varphi) = 0, \quad tr_g R(V, \nabla^{\varphi}_{*} V) d\varphi(*) = 0 \;\; and \;\; tr_g \overset{\varphi}{\nabla}{}^2 V = 0.$$

Lemma 3.3. *Let $\varphi : (M^m, g) \to (N^n, h)$ be a smooth map between the Riemannian manifolds. Then the energy density associated to $V \in \Gamma(\varphi^{-1}TN)$ is given by*

$$e(V) = e(\varphi) + \frac{1}{2} trace_g h(\nabla^{\varphi}_{*} V, \nabla^{\varphi}_{*} V),$$

where $e(\varphi)$ is the energy density of the map φ.

Proof. Let $\{e_1, \ldots, e_m\}$ be a local orthonormal frame on M, then

$$2e(V) = \sum_{i=1}^{m} h^s\big(dV(e_i), dV(e_i)\big).$$

Using Lemma 3.2, we obtain

$$\begin{aligned} 2e(V) &= \sum_{i=1}^{m} h^s\big((dV(e_i))^h, (dV(e_i)^H)\big) + h^s\big((\nabla^{\varphi}_{e_i} V)^V, (\nabla^{\varphi}_{e_i} V)^V\big) \\ &= \sum_{i=1}^{m} h\big(dV(e_i), dV(e_i)\big) + h\big((\nabla^{\varphi}_{e_i} V), (\nabla^{\varphi}_{e_i} V)\big) \\ &= 2e(\varphi) + h\big((\nabla^{\varphi}_{e_i} V), (\nabla^{\varphi}_{e_i} V)\big). \end{aligned}$$

□

Theorem 3.2. *Let $\varphi : (M^m, g) \to (N^n, h)$ be a smooth harmonic map between the Riemannian manifolds and (M, g) be compact. Then the pull-back vector field $V \in \Gamma(\varphi^{-1}TN)$ is harmonic if and only if V is parallel.*

Proof. If φ is harmonic and V is parallel, we deduce that V is harmonic. Conversely, let V_t be a compactly supported variation of V defined by $V = (1 + t)V$. From Lemma 3.3, we have

$$e(V_t) = e(\varphi) + \frac{(1+t)^2}{2} trace_g h(\nabla^{\varphi}_{*} V, \nabla^{\varphi}_{*} V).$$

If V is a critical point of the energy functional, then we have

$$\begin{aligned} 0 &= \frac{d}{dt} E(V_t)_{|t=0} \\ &= \frac{d}{dt}\left(\int_M \Big(e(\varphi) + \frac{(1+t)^2}{2} trace_g h(\nabla^{\varphi} V, \nabla^{\varphi} V)\Big)\, dv_g \right)_{t=0} = \frac{d}{dt} \int_M trace_g h(\nabla^{\varphi} V, \nabla^{\varphi} V)\, dv_g. \end{aligned}$$

It follows that

$$h(\nabla^{\varphi} V, \nabla^{\varphi} V) = 0.$$

□

4 Biharmonicity of pull-back vector fields

In this section, we denote

$$\Delta^{\varphi} V = -trace_g \overset{\varphi}{\nabla}{}^2 V = \sum_{i=1}^{m} \big\{ \nabla^{\varphi}_{\nabla_{e_i} e_i} V - \nabla^{\varphi}_{e_i} \nabla^{\varphi}_{e_i} V \big\}, \tag{4.1}$$

$$S(V) = -\sum_{i=1}^{m} R^N(V, \nabla^{\varphi}_{e_i} V)\, d\varphi(e_i). \tag{4.2}$$

Then we have

$$\tau(V) = \big(\tau(\varphi) - S(V)\big)^H + \big(-\Delta^{\varphi}(V)\big)^V.$$

Theorem 4.1. *Let (M,g) be a compact oriented m-dimensional Riemannian manifold and $V \in \Gamma(\varphi^{-1}TN)$. Then we have*

$$\frac{d}{dt}E_2(V_t)\big|_{t=0} = \int_M \Big\{ h\Big(\Delta^\varphi\Delta^\varphi V + \sum_{i=1}^m \Big[(\nabla^\varphi_{e_i}R)(e_i,S(V))V + R(e_i,\nabla^\varphi_{e_i}S(V))V$$
$$+ 2R(e_i,S(V))\nabla^\varphi_{e_i}V - (\nabla^\varphi_{e_i}R)(e_i,\tau(\varphi))V - R(e_i,\nabla^\varphi_{e_i}\tau(\varphi))V - 2R(e_i,\tau(\varphi))\nabla^\varphi_{e_i}V, V\Big]$$
$$+ h\Big(R(S(V),d\varphi(e_i))d\varphi(e_i) + \Delta^\varphi S(V) - \tau_2(\varphi), v\Big)\Big)\Big\} v_g$$

for any smooth 1-parameter variation $U : M\times(-\epsilon,\epsilon) \xrightarrow{\phi} N \xrightarrow{Y} TN$ of V through vector fields, i.e., $V_t(z) = Y\circ\phi(z,t) = U(z,t) \in T_{\varphi(z)}N$ for any $|t|<\epsilon$ and $z\in M$ or, equivalently, $V_t \in \Gamma(\varphi^{-1}(TN))$ for any $|t|<\epsilon$. Also, W is the tangent vector field on M given by

$$W(z) = \frac{d}{dt}V_z(0), \quad z\in M,$$

where $V_z(t) = U(z,t)$, $(z,t)\in M\times(-\epsilon,\epsilon)$.

Proof. Let $V\in\Gamma(\varphi^{-1}TN)$ and $I=(-\epsilon,\epsilon)$, $\epsilon>0$. For $t\in I$, we denote by $i_t : M\to M\times I$, $p\to(p,t)$, the canonical injection. We consider C^∞-variations $U : M\times I\to TN$ of V, i.e., for all $t\in I$, the mappings $V_t = U\circ i_t$ are, in fact, the vector fields and $V_0 = V$. We choose $\{e_i\}_{i=1}^m$, a local orthonormal frame field of (M,g). We extend e_i (resp. $\frac{d}{dt}\in\Gamma(I)$) to $M\times I$, denoted by E_i (resp. $\frac{d}{dt}$). Moreover, we have $[E_i,\frac{d}{dt}]=0$. We denote by D^ϕ the pull-back Levi–Civita connection of $M\times I$ and by R^D the pull-back Riemann curvature tensor of $M\times I$. Since $M\times I$ is a Riemannian product, we have (using the second Bianchi identity for the last relation)

$$R^D(TN,TI)=0, \quad D^\phi_{\frac{d}{dt}}d\phi(E_i)=0, \quad D^\phi_{E_i}d\phi\Big(\frac{d}{dt}\Big)=0, \quad (D^\phi_{\frac{d}{dt}}R^D)(D^\phi_{E_i}U,U)d\phi(E_i)=0$$

for all $1\le i\le m$. We set

$$Z=\sum_{i=1}^m R^D(D^\phi_{E_i}U,U)\,d\varphi(E_i) \ \text{ and } \ \Omega=\sum_{i=1}^m\big[D^\phi_{D_{E_i}E_i}U - D^\phi_{E_i}D^\phi_{E_i}U\big].$$

We easily observe that $S(V_t)=Z\circ i_t$ and $\Delta^\varphi V_t = \Omega\circ i_t$. In the sequel, we consider the function

$$E_2(V_t) = \frac12\int_M\Big[h(\tau(\varphi),\tau(\varphi)) + h(S(V_t),S(V_t)) - 2h(S(V_t),\tau(\varphi)) + h(\Delta^\varphi V_t,\Delta^\varphi V_t)\Big]v_g$$
$$= \frac12\int_M\Big[h(\tau(\varphi),\tau(\varphi)) + h(Z,Z) - 2h(Z,\tau(\varphi)) + h(\Omega,\Omega)\Big]\circ i_t v_g.$$

Differentiating the function $E_2(V_t)$ at each t, we obtain

$$\frac{d}{dt}E_2(V_t) = \int_M h\big(D^\phi_{\frac{d}{dt}}\tau(\phi),\tau(\phi)\big)\circ i_i v_g + \int_M h(D^\phi_{\frac{d}{dt}}Z,Z)\circ i_i v_g$$
$$- \int_M h(D^\phi_{\frac{d}{dt}}Z,\tau(\phi))\circ i_i v_g + \int_M h(D^\phi_{\frac{d}{dt}}\Omega,\Omega)\circ i_i v_g - \int_M h(Z,D^\phi_{\frac{d}{dt}}\tau(\phi))\circ i_i v_g. \quad (4.3)$$

Taking into account the symmetries of the Riemann curvature tensor and summing over all repeated indices, we have

$$\begin{aligned}
&\int_M h(D^{\phi}_{\frac{d}{dt}} Z, Z) \circ i_t v_g \\
&\quad = \int_M h\Big((D^{\phi}_{\frac{d}{dt}} R^D)(D^{\phi}_{E_i} U, U) d\phi(E_i) + R^D (D^{\phi}_{\frac{d}{dt}} D^{\phi}_{E_i} U, U) d\phi(E_i) + R^D (D^{\phi}_{E_i} U, D^{\phi}_{\frac{d}{dt}} U) d\phi(E_i), Z\Big) \circ i_t v_g \\
&\quad = \int_M \Big[h\Big(R^D (D^{\phi}_{E_i} D^{\phi}_{\frac{d}{dt}} U + \overset{\phi}{R}{}^D\Big(\frac{d}{dt}, E_i\Big) U, U) d\phi(E_i), Z\Big) + h(R^D (d\phi(E_i), Z) D^{\phi}_{E_i} U, D^{\phi}_{\frac{d}{dt}} U)\Big] \circ i_t v_g \\
&\quad = \int_M \Big[- h\big(R^D(d\phi(E_i), Z) U, D^{\phi}_{E_i} D^{\phi}_{\frac{d}{dt}} U\big) + h\big(R^D (d\phi(E_i), Z) D^{\phi}_{E_i} U, D^{\phi}_{\frac{d}{dt}} U\big)\Big] \circ i_t v_g \\
&\quad = \int_M \Big\{ - D^{\phi}_{E_i} \big(h(R^D(d\phi(E_i), Z) U, D^{\phi}_{\frac{d}{dt}} U)\big) + h\big(R^D (D^{\phi}_{E_i} d\phi(E_i), Z) U, D^{\phi}_{\frac{d}{dt}} U\big) \\
&\qquad\qquad + h\big((D^{\phi}_{E_i} R^D)(d\phi(E_i), Z) U, D^{\phi}_{\frac{d}{dt}} U\big) + h\big(R^D (d\phi(E_i), D^{\phi}_{E_i} Z) U, D^{\phi}_{\frac{d}{dt}} U\big) \\
&\qquad\qquad\qquad + 2h\big(R^D (d\phi(E_i), Z) D^{\phi}_{E_i} U, D^{\phi}_{\frac{d}{dt}} U\big)\Big\} \circ i_t v_g. \qquad (4.4)
\end{aligned}$$

Applying the divergence theorem for the 1-form

$$\eta_t(W) = h\big(R(d\varphi(W), S(V_t)) V_t, \nabla^{\varphi}_{\frac{d}{dt}} V_t\big), \ \ t \in I, \ \ d\varphi(W) \in \Gamma(\varphi^{-1}(TN)),$$

the last relation (4.4) gives

$$\begin{aligned}
&\int_M h(D^{\phi}_{\frac{d}{dt}} Z, Z) \circ i_t v_g \\
&\quad = \int_M h\Big((\nabla^{\varphi}_{e_i} R)(d\varphi(e_i), S(V_t)) V_t + R(d\varphi(e_i), \nabla^{\varphi}_{e_i} S(V_t)) V_t + 2R(d\varphi(e_i), S(V_t)) \nabla^{\varphi}_{e_i} V_t, \nabla^{\varphi}_{\frac{d}{dt}} V_t\Big) v_g. \qquad (4.5)
\end{aligned}$$

Similarly, summing over all repeated indices, we deduce

$$\begin{aligned}
&\int_M h(D^{\phi}_{\frac{d}{dt}} \Omega, \Omega) \circ i_t v_g \\
&\quad = \int_M h\big(D^{\phi}_{\frac{d}{dt}} D^{\phi}_{D_{E_i} E_i} U - D^{\phi}_{\frac{d}{dt}} D^{\phi}_{E_i} D^{\phi}_{E_i} U, \Omega\big) \circ i_t v_g = \int_M h\big(D^{\phi}_{D_{E_i} E_i} D^{\phi}_{\frac{d}{dt}} U - D^{\phi}_{E_i} D^{\phi}_{E_i} D^{\phi}_{\frac{d}{dt}} U, \Omega\big) \circ i_t v_g \\
&\quad = \int_M \Big\{ D^{\phi}_{D_{E_i} E_i} \big[h(D^{\phi}_{\frac{d}{dt}} U, \Omega)\big] - h(D^{\phi}_{\frac{d}{dt}} U, D^{\phi}_{D_{E_i} E_i} \Omega) \\
&\qquad\qquad - D^{\phi}_{E_i} \big[h(D^{\phi}_{E_i} D^{\phi}_{\frac{d}{dt}} U, \Omega)\big] + h(D^{\phi}_{E_i} D^{\phi}_{\frac{d}{dt}} U, D^{\phi}_{E_i} \Omega)\Big\} \circ i_t v_g \\
&\quad = \int_M \Big\{ D^{\phi}_{D_{E_i} E_i} \big[h(D^{\phi}_{\frac{d}{dt}} U, \Omega)\big] - D^{\phi}_{E_i} D^{\phi}_{E_i} \big[h(D^{\phi}_{\frac{d}{dt}} U, \Omega)\big] \\
&\qquad\qquad - h(D^{\phi}_{\frac{d}{dt}} U, D^{\phi}_{D_{E_i} E_i} \Omega) + D^{\phi}_{E_i} \big[h(D^{\phi}_{\frac{d}{dt}} U, D^{\phi}_{E_i} \Omega)\big] + h(D^{\phi}_{E_i} D^{\phi}_{\frac{d}{dt}} U, D^{\phi}_{E_i} \Omega)\Big\} \circ i_t v_g \\
&\quad = \Big\{ \Delta^{\phi} \big[h(D^{\phi}_{\frac{d}{dt}} U, \Omega)\big] - h(D^{\phi}_{\frac{d}{dt}} U, D^{\phi}_{D_{E_i} E_i} \Omega) + 2D^{\phi}_{E_i} \big[h(D^{\phi}_{\frac{d}{dt}} U, D^{\phi}_{E_i} \Omega)\big] - h(D^{\phi}_{\frac{d}{dt}} U, D^{\phi}_{E_i} D^{\phi}_{E_i} \Omega)\Big\} \circ i_t v_g \\
&\quad = \Big\{ \Delta^{\phi} \big[h(D^{\phi}_{\frac{d}{dt}} U, \Omega)\big] - h(D^{\phi}_{\frac{d}{dt}} U, D^{\phi}_{D_{E_i} E_i} \Omega) + 2D^{\phi}_{E_i} \big[h(D^{\phi}_{\frac{d}{dt}} U, D^{\phi}_{E_i} \Omega)\big] \\
&\qquad\qquad - 2h(D^{\phi}_{\frac{d}{dt}} U, D^{\phi}_{D_{E_i} E_i} \Omega) + 2h(D^{\phi}_{\frac{d}{dt}} U, D^{\phi}_{D_{E_i} E_i} \Omega) - h(D^{\phi}_{\frac{d}{dt}} U, D^{\phi}_{E_i} D^{\phi}_{E_i} \Omega)\Big\} \circ i_t v_g.
\end{aligned}$$

Applying the divergence Theorem for the 1-form

$$\theta_t(\,\cdot\,) = h(\nabla^{\varphi}_{\frac{d}{dt}} V_t, \nabla^{\varphi}_{\cdot}\Delta^{\varphi} V_t), \;\; t \in I,$$

we have

$$\int_M h(D^{\phi}_{\frac{d}{dt}}\Omega, \Omega) \circ i_t v_g = \int_M \Delta^{\phi}\big[h(D^{\varphi}_{\frac{d}{dt}} V_t, \Delta V_t)\big] v_g + 2\int_M div(\theta_t) v_g + \int_M h(\nabla^{\varphi}_{\frac{d}{dt}} V_t, \Delta^{\varphi}\Delta^{\varphi} V_t) v_g = \int_M h(\nabla^{\varphi}_{\frac{d}{dt}} V_t, \Delta^{\varphi}\Delta^{\varphi} V_t) v_g. \quad (4.6)$$

Similarly, summing over all repeated indices, we deduce

$$\begin{aligned}
&\int_M h(D^{\phi}_{\frac{d}{dt}} Z, \tau(\phi)) \circ i_t v_g \\
&= \int_M h\Big((D^{\phi}_{\frac{d}{dt}} R^D)(D^{\phi}_{E_i} U, U) d\phi(E_i) + R^D(D^{\phi}_{\frac{d}{dt}} D^{\phi}_{E_i} U, U) d\phi(E_i) + R^D(D^{\phi}_{E_i} U, D^{\phi}_{\frac{d}{dt}} U) d\phi(E_i), \tau(\phi)\Big) \circ i_t v_g \\
&= \int_M \Big[h\Big(R^D\Big(D^{\phi}_{E_i} D^{\phi}_{\frac{d}{dt}} U + \overset{\phi}{R}{}^D\Big(\frac{d}{dt}, E_i\Big) U, U\Big) d\phi(E_i), \tau(\phi)\Big) + h\big(R^D(d\phi(E_i), \tau(\phi)) D^{\phi}_{E_i} U, D^{\phi}_{\frac{d}{dt}} U\big)\Big] \circ i_t v_g \\
&= \int_M \Big[-h\big(R^D(d\phi(E_i), \tau(\phi)) U, D^{\phi}_{E_i} D^{\phi}_{\frac{d}{dt}} U\big) + h\big(R^D(d\phi(E_i), \tau(\phi)) D^{\phi}_{E_i} U, D^{\phi}_{\frac{d}{dt}} U\big)\Big] \circ i_t v_g \\
&= \int_M \Big\{ - D^{\phi}_{E_i}\big(h(R^D(d\phi(E_i), \tau(\phi)) U, D^{\phi}_{\frac{d}{dt}} U)\big) + h\big(R^D(D^{\phi}_{E_i} d\phi(E_i), \tau(\phi)) U, D^{\phi}_{\frac{d}{dt}} U\big) \\
&\qquad + h\big((D^{\phi}_{E_i} R^D)(d\phi(E_i), \tau(\phi)) U, D^{\phi}_{\frac{d}{dt}} U\big) + h\big(R^D(d\phi(E_i), D^{\phi}_{E_i}\tau(\phi)) U, D^{\phi}_{\frac{d}{dt}} U\big) \\
&\qquad + 2h\big(R^D(d\phi(E_i), \tau(\phi)) D^{\phi}_{E_i} U, D^{\phi}_{\frac{d}{dt}} U\big)\Big\} \circ i_t v_g.
\end{aligned} \quad (4.7)$$

Applying the divergence Theorem for the 1-form

$$\eta_t(W) = h\big(R(d\varphi(W), \tau(\phi)) V_t, \nabla^{\varphi}_{\frac{d}{dt}} V_t\big), \;\; t \in I, \;\; d\varphi(W) \in \Gamma(\varphi^{-1}(TN))$$

the last relation (4.7) gives

$$\begin{aligned}
&\int_M h(D^{\phi}_{\frac{d}{dt}} Z, \tau(\phi)) \circ i_t v_g \\
&= \int_M h\Big((\nabla^{\varphi}_{e_i} R)(d\varphi(e_i), \tau(\varphi)) V_t + R(d\varphi(e_i), \nabla^{\varphi}_{e_i}\tau(\varphi)) V_t + 2R(d\varphi(e_i), \tau(\varphi)) \nabla^{\varphi}_{e_i} V_t, \nabla^{\varphi}_{\frac{d}{dt}} V_t\Big) v_g.
\end{aligned} \quad (4.8)$$

From Definition 1.1, we have

$$\int_M h(D^{\phi}_{\frac{d}{dt}}\tau(\phi), \tau(\phi))_{|t=0} \circ i_t v_g = -\int_M h(\tau_2(\phi), v) v_g, \quad (4.9)$$

where $v = d\phi(\frac{d}{dt})$ (for more details, see [17]).

Similarly, summing over all repeated indices, we deduce

$$\begin{aligned}
&\int_M h(D^{\phi}_{\frac{d}{dt}}\tau(\phi), Z)\circ i_i v_g \\
&\quad = \int_M h\big(D^{\phi}_{\frac{d}{dt}} Dd\phi(E_i,E_i), Z\big)\circ i_i v_g = \int_M h\big(D^{\phi}_{\frac{d}{dt}} D^{\phi}_{E_i} d\phi(E_i) - D^{\phi}_{\frac{d}{dt}} d\phi(D_{E_i}E_i), Z\big)\circ i_i v_g \\
&= \int_M h\Big(R\Big(d\phi\Big(\frac{d}{dt}\Big), d\phi(E_i)\Big)d\phi(E_i) + D^{\phi}_{E_i} D^{\phi}_{\frac{d}{dt}} d\phi(E_i) + D^{\phi}_{[\frac{d}{dt},E_i]} d\phi(E_i) - D^{\phi}_{D_{E_i}E_i} d\phi\Big(\frac{d}{dt}\Big), Z\Big)\circ i_i v_g \\
&\quad = \int_M \Big(h\Big(R\Big(d\phi\Big(\frac{d}{dt}\Big), d\phi(E_i)\Big)d\phi(E_i), Z\Big) + h\Big(D^{\phi}_{E_i} D^{\phi}_{E_i} d\phi\Big(\frac{d}{dt}\Big), Z\Big)\Big)\circ i_i v_g \\
&\qquad = \int_M \Big(h\Big(R(Z, d\phi(E_i))d\phi(E_i), d\phi\Big(\frac{d}{dt}\Big)\Big) + E_i\Big(h\Big(D^{\phi}_{E_i} d\phi\Big(\frac{d}{dt}\Big), Z\Big)\Big) \\
&\qquad\qquad - E_i\Big(h\Big(d\phi\Big(\frac{d}{dt}\Big), D^{\phi}_{E_i} Z\Big)\Big) + h\Big(D^{\phi}_{E_i} D^{\phi}_{E_i} Z, d\phi\Big(\frac{d}{dt}\Big)\Big)\Big)\circ i_i v_g.
\end{aligned}$$

Applying the divergence Theorem for the 1-form

$$\omega(\,\cdot\,) = \Big(h\Big(D^{\phi}_{\cdot} d\phi\Big(\frac{d}{dt}\Big), Z\Big)\Big), \quad \eta(\,\cdot\,) = h\Big(d\phi\Big(\frac{d}{dt}\Big), D^{\phi}_{\cdot} Z\Big),$$

one gets

$$\int_M h\big(D^{\phi}_{\frac{d}{dt}}\tau(\phi), S(V)\big)\big|_{t=0}\circ i_i v_g = \int_M h\big(R(S(V), d\varphi(e_i))d\varphi(e_i) - \Delta^{\varphi} S(V), v\big) v_g. \tag{4.10}$$

Substituting (4.5), (4.6), (4.8)–(4.10) into (4.3), evaluating at $t=0$ and setting $V = \nabla_{\frac{d}{dt}} V_t|_{t=0}$, we easily obtain the desired result. □

Since the bull-back vector field V is biharmonic if and only if $\frac{d}{dt}\, E_2(V_t)\big|_{t=0} = 0$ for all admissible variations, we get

Corollary. *A pull-back vector field V of an m-dimensional Riemannian manifold (M,g) is biharmonic if and only if*

$$\begin{aligned}
\Delta^{\varphi}\Delta^{\varphi} V + \sum_{i=1}^{m} \Big[&(\nabla^{\varphi}_{e_i} R)(e_i, S(V))V + R(e_i, \nabla^{\varphi}_{e_i} S(V))V \\
&+ 2R(e_i, S(V))\nabla^{\varphi}_{e_i} V - (\nabla^{\varphi}_{e_i} R)(e_i, \tau(\varphi))V - R(e_i, \nabla^{\varphi}_{e_i}\tau(\varphi))V \\
&- 2R(e_i, \tau(\varphi))\nabla^{\varphi}_{e_i} V + R(S(V), d\varphi(e_i))d\varphi(e_i) + \Delta^{\varphi} S(V) - \tau_2(\varphi)\Big] = 0.
\end{aligned}$$

Remark. If a pull-back vector field of a Riemannian manifold (M,g) defines a harmonic map from (M,g) into (TN, h^s), i.e., $S(V)=0$, $\tau(\varphi)=0$ and $\Delta^{\varphi} V = 0$, then it is automatically a biharmonic pull-back vector field.

Theorem 4.2. *Let $\varphi : (M^n, g) \to (N^m, h)$ be a smooth harmonic map between the Riemannian manifolds and (M,g) be compact. Then the pull-back bundle $V \in \Gamma(\varphi^{-1}TN)$ is biharmonic if and only if V is harmonic.*

Proof. Let V_t be a compactly supported variation of V defined by $V_t = (1+t)V$. From formulas (4.1) and (4.2), we have

$$\Delta^\varphi V_t = (1+t)\Delta^\varphi V,$$
$$S(V_t) = (1+t)^2 S(V),$$
$$E_2(V_t) = \frac{1}{2}\int\limits_M h^s(\tau(V_t),\tau(V_t))v_g = \frac{1}{2}\int\limits_M h(\Delta^\varphi V_t,\Delta^\varphi V_t)v_g + \frac{1}{2}\int\limits_M h(S(V_t),S(V_t))v_g$$
$$= \frac{(1+t)^2}{2}\int\limits_M h(\Delta^\varphi V,\Delta^\varphi V)v_g + \frac{(1+t)^4}{2}\int\limits_M h(S(V_t),S(V_t))v_g.$$

Since the pull-back vector field V is biharmonic, then for the variation V_t, we have

$$\frac{d}{dt}E_2(V_t)\big|_{t=0} = \int\limits_M h(\Delta^\varphi V,\Delta^\varphi V)v_g + 2\int\limits_M h(S(V_t),S(V_t))v_g = 0.$$

Hence

$$\Delta^\varphi V = 0 \text{ and } S(V_t) = 0,$$

thus V is harmonic and Theorem 3.2 follows. □

Example. We give in $\mathbb{R}^3$ the curve $\gamma : [0,2\pi] \to \mathbb{R}^3$, $t \mapsto \gamma(t) = (t,t,t)$. Let $V_0 = (1,2,-1)$ and ∇ be the connection on $\mathbb{R}^3$ such that $\Gamma^1_{12} = x$ and the other coefficients are zero. We propose to calculate the vector field $V(t) \in \Gamma(\gamma^{-1}T\mathbb{R}^3)$ which is parallel and extends V_0.

The general form of a vector field $V(t) \in \Gamma(\gamma^{-1}T\mathbb{R}^3)$ is

$$V = v^1(t)\frac{\partial}{\partial x} + v^2(t)\frac{\partial}{\partial y} + v^3(t)\frac{\partial}{\partial z}.$$

Then V is biharmonic if it verifies the following system:

$$\begin{cases} \dfrac{dv_1(t)}{dt} + v^2(t)\dfrac{dc^1(t)}{dt}\Gamma^1_{12} = 0, \\ \dfrac{dv^2(t)}{dt} = 0, \\ \dfrac{dv^3(t)}{dt} = 0, \end{cases}$$

thus

$$\begin{cases} \dfrac{dv_1(t)}{dt} + v^2(t)t = 0, \\ v^2(t) = b, \\ v^3(t) = c, \end{cases}$$

where b and c are arbitrary real constants. Hence

$$\begin{cases} v_1(t) = -b\dfrac{t^2}{2} + d, \\ v^2(t) = b, \\ v^3(t) = c. \end{cases}$$

Since $(V)_{t=0} = V_0$, we have

$$V = (-t^2+1)\frac{\partial}{\partial x} + 2\frac{\partial}{\partial y} - \frac{\partial}{\partial z}.$$

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