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JUSTIFICATION OF THE QUASI-STATIC CASE IN FRICTIONLESS PROBLEMS OF THE COUPLE-STRESS THEORY OF VISCOELASTICITY

Abstract. We consider three-dimensional boundary value problems in the couple-stress theory of viscoelasticity and establish that, in a certain well-defined sense, the solution of the dynamic problem for inhomogeneous anisotropic media and the solution of the corresponding quasi-static problem are “close” to each other. Weak solutions of the problems are studied in both the coercive and non-coercive cases. It is proved that when some of the data of the problem do not depend on time, the solutions of the dynamic and quasi-static problems are “close”, provided that friction is not taken into account.

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1 Introduction

The general and widespread use of the linear theory of viscoelasticity is observed in the seventies of the past century. Activity in this area is associated with an extensive application of polymeric materials with properties that are obviously not described by either elastic or viscous models, but combine the features of both models. Strictly grounded mathematical theory of linear viscoelasticity, with numerous practical applications, is contained in the monographs of D. R. Bland and R. M. Christensen (see [1,2] and the references therein).

Viscoelastic materials are those supplied with "memory" in the sense that the state at time t depends on all the deformations that the material undergoes. A particularly important class of "viscoelastic equations of state" is associated with materials for which there is a linear relationship between the time derivatives of the stress and strain tensors. We will consider viscoelastic materials with short-term memory, i.e., when the stress at time t depends only on the deformations, the moment at time t and the nearest preceding moments of time. In the considered model of the theory of elasticity, unlike classical theory, every elementary medium particle undergoes both displacement and rotation. In this case, all mechanical values are expressed by the displacement and rotation vectors. In their work [4], E. Cosserat and F. Cosserat created and presented the model of a solid medium in which every material point has six degrees of freedom, defined by three displacement components and three components of rotation (for the history of the model, see [6,26,29,32] and the references therein). The basic equations of the model are interrelated and generate a second order matrix differential operator of dimension 6×6. The basic boundary value problems and also the transmission problems of the hemitropic theory of elasticity for smooth and non-smooth Lipschitz domains were studied in [30]. The one-sided contact problems of statics of the hemitropic theory of elasticity free from friction were investigated in [13, 14, 16, 18, 23], the contact problems of statics and dynamics with a friction were considered in [9–12,15,17,19–22], and the one-sided problems of classical linear theory of elasticity were considered in many papers and monographs (see [5,7,8,24,25] and the references therein). Particular problems of the viscoelasticity theory are considered in [1,2]. As for the dynamical and quasistatical boundary-contact problems of viscoelasticity with friction, we refer to [5].

The paper is organized as follows. First, we present the general field equations of the linear theory of couple-stress viscoelasticity; Green's formula and positive-definite forms of the potential energy are written out. Next, Dynamic and quasi-static frictionless problems are formulated, and two theorems are presented proving the "closeness" of the solutions of these problems to the solution of a certain static problem. Both the coercive and non-coercive cases are considered.

2 Field equations and Green's formulas

2.1 Basic equations

Let $\Omega \subset \mathbb{R}^3$ be a bounded, simply connected domain with a C^∞-smooth boundary $S := \partial\Omega$, $\overline{\Omega} = \Omega \cup S$. Throughout the paper, $n(x) = (n_1(x), n_2(x), n_3(x))$ denotes the outward unit normal vector at a point $x \in S$. Assume that the region Ω is filled with an inhomogeneous, anisotropic viscoelastic material.

The basic equilibrium equations of dynamics of couple-stress viscoelasticity for inhomogeneous anisotropic bodies read as

$$\begin{aligned} \partial_i \sigma_{ij}(x,t) + \varrho\, F_j(x,t) &= \varrho\, \frac{\partial^2 u_j(x,t)}{\partial t^2}\,, \\ \partial_i \mu_{ij}(x,t) + \varepsilon_{ikj}\sigma_{ik}(x,t) + \varrho\, G_j(x,t) &= \mathcal{J}\, \frac{\partial^2 \omega_j(x,t)}{\partial t^2}\,, \end{aligned} \tag{2.1}$$

where t is the time variable, $\partial = (\partial_1, \partial_2, \partial_3)$ with $\partial_i = \frac{\partial}{\partial x_i}$, ϱ is the mass density of the elastic material, $\mathcal{J}$ is the moment of inertia per unit volume, $F = (F_1, F_2, F_3)^\top$ and $G = (G_1, G_2, G_3)^\top$ are the body force and body couple vectors per unit mass, $u = (u_1, u_2, u_3)^\top$ is the *displacement vector*, $\omega = (\omega_1, \omega_2, \omega_3)^\top$ is the *micro-rotation vector*, ε_{ikj} is the permutation (Levi–Civita) symbol; here and in what follows, the symbol $(\,\cdot\,)^\top$ denotes transposition, and summation over repeated indices is meant from 1 to 3. For the *force stress tensor* $\{\sigma_{ij}\}$ and the *couple stress tensor* $\{\mu_{ij}\}$, we have

$$\begin{aligned}\sigma_{ij}(x,t) &:= \sigma_{ij}(U(t))\\ &= a^{(0)}_{ijlk}(x)\zeta_{lk}(U(t)) + b^{(0)}_{ijlk}(x)\eta_{lk}(U(t)) + a^{(1)}_{ijlk}(x)\partial_t\zeta_{lk}(U(t)) + b^{(1)}_{ijlk}(x)\partial_t\eta_{lk}(U(t)),\\ \mu_{ij}(x,t) &:= \mu_{ij}(U(t))\\ &= b^{(0)}_{ijlk}(x)\zeta_{lk}(U(t)) + c^{(0)}_{ijlk}(x)\eta_{lk}(U(t)) + b^{(1)}_{ijlk}(x)\partial_t\zeta_{lk}(U(t) + c^{(1)}_{ijlk}(x)\partial_t\eta_{lk}(U(t)),\end{aligned}$$

where $U(t) := U(x,t) = (u(x,t),\omega(x,t))^\top$, $\zeta_{lk}(U(t)) = \partial_l u_k(x,t) - \varepsilon_{lkm}\omega_m(x,t)$ and $\eta_{lk}(U(t)) = \partial_l\omega_k(x,t)$ are the so-called strain and torsion (curvature) tensors; the real-valued functions $a^{(0)}_{ijlk}$, $b^{(0)}_{ijlk}$, $c^{(0)}_{ijlk}$ (resp., $a^{(1)}_{ijlk}$, $b^{(1)}_{ijlk}$, $c^{(1)}_{ijlk}$), called the elastic constants (resp., the viscosity constants), satisfy some smoothness and symmetry conditions:

(i) $a^{(q)}_{ijlk}, b^{(q)}_{ijlk}, c^{(q)}_{ijlk} \in C^1(\overline{\Omega})$,

(ii) $a^{(q)}_{ijlk} = a^{(q)}_{lkij}$, $c^{(q)}_{ijlk} = c^{(q)}_{lkij}$,

(iii) there exists $\alpha_0 > 0$ such that $\forall\, x \in \overline{\Omega}$ and $\forall\, \xi_{ij}, \eta_{ij} \in R$:

$$a^{(q)}_{ijlk}(x)\xi_{ij}\xi_{lk} + 2b^{(q)}_{ijlk}(x)\xi_{ij}\eta_{lk} + c^{(q)}_{ijlk}(x)\eta_{ij}\eta_{lk} \geq \alpha_0(\xi_{ij}\xi_{ij} + \eta_{ij}\eta_{ij}),\ \ q = 0,1.$$

We introduce a matrix differential operator corresponding to the left-hand side expressions of system (2.1):

$$\mathcal{M}(x,\partial) = \begin{bmatrix}\mathcal{M}^{(1)}(x,\partial) & \mathcal{M}^{(2)}(x,\partial)\\ \mathcal{M}^{(3)}(x,\partial) & \mathcal{M}^{(4)}(x,\partial)\end{bmatrix}_{6\times 6},\quad \mathcal{M}^{(p)}(x,\partial) = \Big[\mathcal{M}^{(p)}_{jk}(x,\partial)\Big]_{3\times 3},\ \ p = \overline{1,4}$$

where

$$\begin{aligned}\mathcal{M}^{(1)}_{jk}(x,\partial) &= \partial_i\big(\big[a^{(0)}_{ijlk}(x) + a^{(1)}_{ijlk}(x)\partial_t\big]\partial_l\big),\\ \mathcal{M}^{(2)}_{jk}(x,\partial) &= \partial_i\big(\big[b^{(0)}_{ijlk}(x) + b^{(1)}_{ijlk}(x)\partial_t\big]\partial_l\big) - \varepsilon_{lrk}\partial_i\big[a^{(0)}_{ijlr}(x) + a^{(1)}_{ijlr}(x)\partial_t\big];\\ \mathcal{M}^{(3)}_{jk}(x,\partial) &= \partial_i\big(\big[b^{(0)}_{lkij}(x) + b^{(1)}_{lkij}(x)\partial_t\big]\partial_l\big) + \varepsilon_{irj}\big[a^{(0)}_{irlk}(x) + a^{(1)}_{irlk}(x)\partial_t\big]\partial_l;\\ \mathcal{M}^{(4)}_{jk}(x,\partial) &= \partial_i\big(\big[c^{(0)}_{ijlk}(x) + c^{(1)}_{ijlk}(x)\partial_t\big]\partial_l\big) - \varepsilon_{lrk}\partial_i\big[b^{(0)}_{lrij}(x) + b^{(1)}_{lrij}(x)\partial_t\big]\\ &\quad + \varepsilon_{irj}[b^{(0)}_{irlk}(x) + b^{(1)}_{irlk}(x)\partial_t]\partial_l - \varepsilon_{ipj}\varepsilon_{lrk}[a^{(0)}_{iplr}(x) + a^{(1)}_{iplr}(x)\partial_t].\end{aligned}$$

Denote by $\mathcal{N}(\partial,n)$ the generalized 6×6 matrix differential stress operator

$$\mathcal{N}(\partial,n) = \begin{bmatrix}\mathcal{N}^{(1)}(\partial,n) & \mathcal{N}^{(2)}(\partial,n)\\ \mathcal{N}^{(3)}(\partial,n) & \mathcal{N}^{(4)}(\partial,n)\end{bmatrix}_{6\times 6},\quad \mathcal{N}^{(p)}(\partial,n) = \Big[\mathcal{N}^{(p)}_{jk}(\partial,n)\Big]_{3\times 3},\ \ p = \overline{1,4},$$

where

$$\begin{aligned}\mathcal{N}^{(1)}_{jk}(\partial,n) &= \big[a^{(0)}_{ijlk} + a^{(1)}_{ijlk}\partial_t\big]n_i\partial_l;\\ \mathcal{N}^{(2)}_{jk}(\partial,n) &= \big[b^{(0)}_{ijlk} + b^{(1)}_{ijlk}\partial_t\big]n_i\partial_l - \varepsilon_{lrk}\big[a^{(0)}_{ijlr} + a^{(1)}_{ijlr}\partial_t\big]n_i;\\ \mathcal{N}^{(3)}_{jk}(\partial,n) &= \big[b^{(0)}_{lkij} + b^{(1)}_{lkij}\partial_t\big]n_i\partial_l;\\ \mathcal{N}^{(4)}_{jk}(\partial,n) &= \big[c^{(0)}_{ijlk} + c^{(1)}_{ijlk}\partial_t\big]n_i\partial_l - \varepsilon_{lrk}\big[b^{(0)}_{lrij} + b^{(1)}_{lrij}\partial_t\big]n_i.\end{aligned}\tag{2.2}$$

Here, $\partial_n = \partial/\partial n$ denotes the directional derivative along the vector n (normal derivative). In the sequel, for the force stress and couple stress vectors, we use the following notation:

$$\mathcal{T}U = \mathcal{N}^{(1)}u + \mathcal{N}^{(2)}\omega,\quad MU = \mathcal{N}^{(3)}u + \mathcal{N}^{(4)}\omega,$$

where $\mathcal{N}^{(p)}$, $p = 1,2,3,4$, are defined by (2.2).

The system of equations (2.1) can be rewritten in the matrix form

$$\mathcal{M}(x,\partial)U(x,t)+\mathcal{G}(x,t)=P\,\frac{\partial^2 U(x,t)}{\partial t^2}\,,\quad x\in\Omega,\ \ 0<t<T, \tag{2.3}$$

where T is an arbitrary positive number, $U=(u,\omega)^\top$, $\mathcal{G}=(\varrho\,F,\varrho\,G)^\top$, $P=[p_{ij}]_{6\times 6}$, $p_{ii}=\varrho$, when $i=1,2,3$, $p_{ii}=\mathcal{J}$, when $i=4,5,6$, and $p_{ij}=0$, when $i\neq j$.

Throughout the paper, $L_p(\Omega)$ $(1\le p\le\infty)$, $L_2(\Omega)=H^0(\Omega)$ and $H^s(\Omega)=H^s_2(\Omega)$, $s\in\mathbb{R}$, denote the Lebesgue and Bessel potential spaces (see, e.g., [27,33]). We denote the corresponding norms by the symbols $\|\cdot\|_{L_p(\Omega)}$ and $\|\cdot\|_{H^s(\Omega)}$. Denote by $D(\Omega)$ the class of $C^\infty(\Omega)$ functions with support in the domain Ω. If M is an open proper part of the manifold $\partial\Omega$, i.e., $M\subset\partial\Omega$, $M\neq\partial\Omega$, then we denote by $H^s(M)$ the restriction of the space $H^s(\partial\Omega)$ on M:

$$H^s(M):=\big\{r_{_M}\varphi:\ \varphi\in H^s(\partial\Omega)\big\},$$

where $r_{_M}$ stands for the restriction operator on the set M. Further, let

$$\widetilde{H}^s(M):=\big\{\varphi\in H^s(\partial\Omega):\ \operatorname{supp}\varphi\subset\overline{M}\big\}.$$

The total strain energy of the respective media has the form

$$\mathcal{B}^{(q)}(U,V)=\int\limits_{\Omega}\Big\{a^{(q)}_{ijlk}(x)\zeta_{ij}(U)\zeta_{lk}(V)+b^{(q)}_{ijlk}(x)\zeta_{ij}(U)\eta_{lk}(V)$$
$$+b^{(q)}_{ijlk}(x)\zeta_{ij}(V)\eta_{lk}(U)+c^{(q)}_{ijlk}(x)\eta_{ij}(U)\eta_{lk}(V)\Big\}\,dx,$$

where $q=1,2$, $U=(u,\omega)^\top$, $V=(v,w)^\top$ and $\zeta_{ij}(U)=\partial_i u_j-\varepsilon_{ijr}\omega_r$, $\eta_{ij}(U)=\partial_i\omega_j$.

From the properties (ii) and (iii), it is clear that $\mathcal{B}^{(q)}(U,V)=\mathcal{B}^{(q)}(V,U)$ and $\mathcal{B}^{(q)}(U,U)\ge 0$. Moreover, there exist positive constants C_1 and C_2, depending only on the material parameters, such that Korn's type inequality (cf. [8, Part I, § 12], [3, § 6.3])

$$\mathcal{B}^{(q)}(U,U)\ge C_1\|U\|^2_{[H^1(\Omega)]^6}-C_2\|U\|^2_{[L_2(\Omega)]^6},\quad q=1,2 \tag{2.4}$$

holds for an arbitrary real-valued vector function $U\in[H^1(\Omega)]^6$.

Remark 2.1. If $U\in[H^1(\Omega)]^6$ and on some open part $S^*\subset\partial\Omega$ the trace $\{U\}^+$ vanishes, i.e., $r_{_{S^*}}\{U\}^+=0$, then we have the strict Korn's inequality

$$\mathcal{B}^{(q)}(U,U)\ge c\|U\|^2_{[H^1(\Omega)]^6}$$

with some positive constant $c>0$ which does not depend on the vector U. This follows from (2.4) and the fact that in this case $\mathcal{B}^{(q)}(U,U)>0$ for $U\neq 0$ (see [31], [28, Chapter 2, Exercise 2.17]).

2.2 Green's formulas

For the real-valued vector functions $U(t)=(u(t),\omega(t))^\top$ and $\widetilde{U}(t)=(\widetilde{u}(t),\widetilde{\omega}(t))^\top$ from the class $[C^2(\overline{\Omega})]^6$ and for an arbitrary $t\in[0;T]$, the following Green formula holds (see [15]):

$$\int\limits_{\Omega}\mathcal{M}(\partial)U(t)\cdot\widetilde{U}(t)\,dx$$
$$=\int\limits_{S}\{\mathcal{N}(\partial,n)U(t)\}^+\cdot\{\widetilde{U}(t)\}^+\,dS-\big\{\mathcal{B}^{(0)}(U(t),\widetilde{U}(t))+\partial_t\mathcal{B}^{(1)}(U(t),\widetilde{U}(t))\big\}, \tag{2.5}$$

where $\{\,\cdot\,\}^+$ denotes the trace operator on S from Ω.

By standard limiting arguments, Green's formula (2.5) can be extended to Lipschitz domains and to vector functions $U, \widetilde{U} \in [H^1(\Omega)]^6$ with $\mathcal{M}(x,\partial)U(t) \in [L_2(\Omega)]^6$ (see [27,31]):

$$\int_{\Omega} \mathcal{M}(\partial)U(t) \cdot \widetilde{U}(t)\,dx = \big\langle \{\mathcal{N}(\partial,n)U(t)\}^+ \cdot \{\widetilde{U}(t)\}^+ \big\rangle_s \,dS - \big\{\mathcal{B}^{(0)}(U(t),\widetilde{U}(t)) + \partial_t \mathcal{B}^{(1)}(U(t),\widetilde{U}(t))\big\}, \ \ t \in (0;T),$$

where $\langle \cdot\,,\,\cdot \rangle_S$ denotes the duality between the spaces $[H^{-1/2}(S)]^6$ and $[H^{1/2}(S)]^6$, that generalizes the usual inner product in the space $[L_2(\partial\Omega)]^6$. By this relation, the generalized trace of the stress operator $\{\mathcal{N}(\partial,n)U\}^+ \in [H^{-1/2}(S)]^6$ is well defined.

The following assertion describes the null space of the energy quadratic form $\mathcal{B}^{(q)}(U(t),U(t))$ (see [15]).

Lemma 2.1. *Let $U(t) = (u(t),\omega(t))^\top \in [C^1(\overline{\Omega})]^6$ and $\mathcal{B}^{(q)}(U(t),U(t)) = 0$ in Ω for arbitrary $t \in (0;T)$. Then*

$$u(t) = [a^{(q)} \times x] + b^{(q)}, \ \ \omega(t) = a^{(q)}, \ \ x \in \Omega,$$

where $a^{(q)}$ and $b^{(q)}$ are arbitrary three-dimensional constant vectors and the symbol $[\,\cdot \times \cdot\,]$ denotes the cross product of two vectors.

Vectors of type $([a^{(q)} \times x] + b^{(q)}, a^{(q)})$ are called *generalized rigid displacement vectors.* Note that a generalized rigid displacement vector vanishes, i.e., $a^{(q)} = b^{(q)} = 0$ if it is zero at a single point.

3 Contact problems without friction

3.1 Pointwise and variational formulation of the contact problem

Let X be a Banach space with the norm $\|\cdot\|_X$. We denote by $L_p(0,T;X)(1 \le p \le \infty)$ the space of measurable functions $t \mapsto f(t)$ defined on the interval $(0;T)$ with values in the space X such that

$$\|f\|_{L_p(0,T;X)} := \Big\{ \int_0^T \|f(t)\|_X^p \,dt \Big\}^{1/p} < \infty \ \ \text{for} \ \ 1 \le p < \infty$$

and

$$\|f\|_{L_\infty(0,T;X)} := \operatorname*{ess\,sup}_{t\in(0;T)} \{\|f(t)\|_X\} < \infty \ \ \text{for} \ \ p = \infty\,.$$

Definition 3.1. The vector-function $U : (0;T) \to [H^1(\Omega)]^6$ is said to be a weak solution of equation (2.3) for $\mathcal{G} : (0;T) \to [L_2(\Omega)]^6$ if

$$U(t), U'(t) \in L_\infty(0,T;[H^1(\Omega)]^6), \quad U''(t) \in L_\infty(0,T;[L_2(\Omega)]^6),$$

and for every $\Phi \in [\mathcal{D}(\Omega)]^6$,

$$(PU''(t),\Phi) + \mathcal{B}^{(0)}(U(t),\Phi) + \mathcal{B}^{(1)}(U'(t),\Phi) = (\mathcal{G}(t),\Phi).$$

Here and in what follows, the symbol $(\,\cdot\,,\,\cdot\,)$ denotes the scalar product in the space $[L_2(\Omega)]^6$.

Let $S = S_1 \cup S_2$, $\operatorname{mes} S_i > 0, i = 1,2$, $S_1 \cap S_2 = \varnothing$, $\overline{S}_1 \cup \overline{S}_2 = S$.

Further, let

$$\mathcal{G} : (0,T) \to [L_2(\Omega)]^6, \ \ \varphi : (0;T) \to [H^{-1/2}(S_2)]^3, \ \ f : (0;T) \to L_\infty(S_2).$$

Consider the following contact problem of dynamics without friction.

Problem (A). Find a weak solution $U : (0;T) \to [H^1(\Omega)]^6$ of the equation

$$\mathcal{M}(x,\partial)U(x,t) + \mathcal{G}(x,t) = P\,\frac{\partial^2 U(x,t)}{\partial t^2}\,, \ \ x \in \Omega, \ \ t \in (0;T),$$

satisfying the following initial conditions:

$$U(x,0)=0,\quad x\in\Omega, \tag{3.1}$$
$$U'(x,0)=0,\quad x\in\Omega, \tag{3.2}$$

and the boundary contact conditions:

$$r_{S_1}\{U\}^+=0 \ \text{ on } \ S_1\times(0;T), \tag{3.3}$$
$$r_{S_2}\{(\mathcal{T}U)_n\}^+=f \ \text{ on } \ S_2\times(0;T), \tag{3.4}$$
$$r_{S_2}\{MU\}^+=\varphi \ \text{ on } \ S_2\times(0;T), \tag{3.5}$$
$$r_{S_2}\{(\mathcal{T}U)_s\}^+=0 \ \text{ on } \ S_2\times(0;T). \tag{3.6}$$

This problem can be reformulated in terms of a variational inequality. To this end, we introduce the closed convex sets $\mathcal{K}$ and $\mathcal{K}_0$:

$$\mathcal{K}:=\Big\{V:\ \ V(t),V'(t)\in L_\infty(0,T;[H^1(\Omega)]^6),\ \ V''(t)\in L_\infty(0,T;[L_2(\Omega)]^6),$$
$$r_{S_1}\{V\}^+=0,\ \ V(0)=V'(0)=0\Big\};$$
$$\mathcal{K}_0:=\big\{V:\ V\in[H^1(\Omega)]^6,\ \ r_{S_1}\{V\}^+=0\big\}.$$

Consider the following variational inequality: Find a vector-function $U=(u,\omega)^\top\in\mathcal{K}$ such that the variational inequality

$$(PU''(t),V-U'(t))+\mathcal{B}^{(0)}(U(t),V-U'(t))+\mathcal{B}^{(1)}(U'(t),V-U'(t))$$
$$\geq(\mathcal{G}(t),V-U'(t))+\int\limits_{S_2}f(t)\{v_n-u_n'(t)\}^+\,dS+\big\langle\varphi(t),r_{S_2}\{w-\omega'(t)\}^+\big\rangle_{S_2} \tag{3.7}$$

holds for all $V=(v,w)^\top\in\mathcal{K}_0$.

Here and in what follows, the symbol $\langle\,\cdot\,,\,\cdot\,\rangle$ denotes the duality relation between the corresponding dual pairs $X^*(M)$ and $X(M)$. In particular, $\langle\,\cdot\,,\,\cdot\,\rangle_{S_2}$ in (3.7) denotes the duality relation between the spaces $[H^{-1/2}(S_2)]^3$ and $[\widetilde{H}^{1/2}(S_2)]^3$.

In [15], it is proved (Theorem 4.1) that Problem A and variational inequality (3.7) are equivalent. Moreover, each solution $U:(0;T)\to[H^1(\Omega)]^6$ of Problem A is a solution of the variational inequality (3.7), and vice versa.

It is easy to show that the variational inequality (3.7) is equivalent to the following variational equality: Find $U:(0;T)\to[H^1(\Omega)]^6$ satisfying the following variational identity:

$$(PU''(t),V)+\mathcal{B}^{(0)}(U(t),V)+\mathcal{B}^{(1)}(U'(t),V)=(\mathcal{G}(t),V)+\int\limits_{S_2}f(t)\{v_n\}^+\,dS+\big\langle\varphi(t),r_{S_2}\{w\}^+\big\rangle_{S_2} \tag{3.8}$$

for all $V\in\mathcal{K}_0$.

Indeed, if instead of V in inequality (3.7) we take $U'(t)+\lambda W\in\mathcal{K}_0$ with arbitrary $\lambda>0$ and $W=(\xi,\eta)^\top\in\mathcal{K}_0$, we obtain

$$\lambda(PU''(t),W)+\lambda\mathcal{B}^{(0)}(U(t),W)+\lambda\mathcal{B}^{(1)}(U'(t),W)$$
$$\geq\lambda(\mathcal{G}(t),W)+\lambda\int\limits_{S_2}f(t)\{\xi_n\}^+\,dS+\lambda\big\langle\varphi(t),r_{S_2}\{\eta\}^+\big\rangle_{S_2} \tag{3.9}$$

for all $W\in\mathcal{K}_0$. Dividing both sides of inequality (3.9) by λ and taking $-W$ instead of W in (3.9), we obtain the opposite inequality, which together with (3.9), yields equality (3.8). To shorten the notation, we denote the right-hand side of inequality (3.9) by (L,W), i.e.,

$$(L,W):=(\mathcal{G}(t),W)+\int\limits_{S_2}f(t)\{\xi_n\}^+\,dS+\big\langle\varphi(t),r_{S_2}\{\eta\}^+\big\rangle_{S_2}\ \text{ for all }\ W\in\mathcal{K}_0.$$

Then (3.8) takes the form

$$(PU''(t),V)+\mathcal{B}^{(0)}(U(t),V)+\mathcal{B}^{(1)}(U'(t),V)=(L,V),\ \ \forall\, V\in\mathcal{K}_0. \tag{3.10}$$

We consider a dynamic problem with zero initial conditions and assume that the data $\mathcal{G}$, f and φ do not depend on t. Along with this, we consider a quasi-static problem without friction:

Find $\widetilde{U}:(0;T)\to[H^1(\Omega)]^6$, satisfying the following variational equality:

$$\mathcal{B}^{(0)}(\widetilde{U}(t),V)+\mathcal{B}^{(1)}(\widetilde{U}'(t),V)=(L,V),\ \ V\in\mathcal{K}_0,\ \ \widetilde{U}\in\mathcal{K}_0, \tag{3.11}$$

and the initial condition $\widetilde{U}(0)=0$.

Our goal is to show that the solution $U(t)$ of the dynamic problem (3.10) and the solution $\widetilde{U}(t)$ of the quasi-static problem (3.11) are "close" in a certain sense (which will be explained below).

We will consider separately two cases: $\operatorname{mes} S_1>0$ and $S_1=\varnothing$.

4 The case $\operatorname{mes} S_1>0$

First of all, we consider the case $\operatorname{mes} S_1>0$ and compare the solutions $U(t)$ and $\widetilde{U}(t)$ with the solution of the following static problem:

Find $\overline{U}\in[H^1(\Omega)]^6$ satisfying the equation

$$\mathcal{B}^{(0)}(\overline{U},V)=(L,V),\ \ \forall\, V\in\mathcal{K}_0,\ \ \overline{U}\in\mathcal{K}_0. \tag{4.1}$$

Note that this problem has a unique solution when $\operatorname{mes} S_1>0$. Indeed, let $\overline{U}_1$ and $\overline{U}_2$ be two solutions of problem (4.1). Then the difference $W=\overline{U}_1-\overline{U}_2\in\mathcal{K}_0$ satisfies the equality

$$\mathcal{B}^{(0)}(W,V)=0,\ \ \forall\, V\in\mathcal{K}_0.$$

Therefore, $\mathcal{B}^{(0)}(W,W)=0$. Since $\operatorname{mes} S_1>0$, the bilinear form $\mathcal{B}^{(0)}$ is strictly coercive on $\mathcal{K}_0$ and $\mathcal{B}^{(0)}(W,W)\geq\beta\|W\|^2_{[H^1(\Omega)]^6}$, $\forall\, W\in\mathcal{K}_0$, with some $\beta>0$. Hence, we find that $W=0$, i.e., $\overline{U}_1=\overline{U}_2$.

We formulate two theorems from which the final result on the "closeness" of solutions of dynamic and quasi-static problems follows.

Theorem 4.1. *Let $\operatorname{mes} S_1>0$ and let $\mathcal{G}$, f, φ do not depend on t. Then, under conditions (3.1), (3.2), (3.3)–(3.6), there exist positive constants $\gamma>0$ and $C>0$ such that*

$$\|U-\overline{U}\|_{[H^1(\Omega)]^6}\leq Ce^{-\gamma t}, \tag{4.2}$$

$$\|U'(t)\|_{[L_2(\Omega)]^6}\leq Ce^{-\gamma t},\ \ e^{\gamma t}U'\in L_2(0,\infty;[H^1(\Omega)]^6), \tag{4.3}$$

where $\overline{U}\in[H^1(\Omega)]^6$ is the solution of the static problem (4.1), and U is the solution of the dynamic problem without friction.

Theorem 4.2. *Let $\operatorname{mes} S_1>0$. Then, under the conditions of Theorem 4.1, there exist positive constants $\gamma>0$ and $C>0$ such that*

$$\|\widetilde{U}-\overline{U}\|_{[H^1(\Omega)]^6}\leq Ce^{-\gamma t},\ \ e^{\gamma t}\widetilde{U}'\in L_2(0,\infty;[H^1(\Omega)]^6), \tag{4.4}$$

where $\widetilde{U}$ is the solution of the quasi-static problem (3.11) without friction satisfying the initial condition $\widetilde{U}(0)=0$.

Corollary 4.1 (Justification of the quasi-static case)**.** *Under the conditions of Theorem 4.1, there exist positive constants $\gamma>0$ and $C>0$ such that*

$$\|U(t)-\widetilde{U}(t)\|_{[H^1(\Omega)]^6}\leq Ce^{-\gamma t},\ \ e^{\gamma t}(U'-\widetilde{U}')\in L_2(0,\infty;[H^1(\Omega)]^6).$$

Before proceeding to the proofs of the above-mentioned theorems, we specify conditions that are satisfied by the solution $U(t)$ of the dynamic problem without friction, the solution $\overline{U}$ of the stationary problem, and their difference $W(t)=U(t)-\overline{U}$.

(i) $U(t)$ satisfies equation (3.10) and the initial conditions $U(0) = U'(0) = 0$;

(ii) $\overline{U}$ satisfies equation (4.1) ($\overline{U}' = \overline{U}'' = 0$, since $\overline{U}$ does not depend on t due to the stationarity of the problem);

(iii) $W(t) = U(t) - \overline{U}$ satisfies the equation

$$(PW''(t), V) + \mathcal{B}^{(1)}(W'(t), V) + \mathcal{B}^{(0)}(W(t), V) = 0 \tag{4.5}$$

with the following initial conditions:

$$W'(0) = 0, \quad W(0) = -\overline{U}. \tag{4.6}$$

The validity of equation (4.5) with the initial conditions (4.6) becomes clear if we take into account that $U(t)$ is a solution of the variational equation (3.10) with zero initial conditions and $\overline{U}$ is a solution of the static equation (4.1) not depending on t.

For some $\lambda > 0$, we introduce the function $W(t) = e^{-\lambda t}Z(t)$, where $Z(t) = e^{\lambda t}(U(t) - \overline{U}$. Then $W'(t) = e^{-\lambda t}(Z'(t) - \lambda Z(t))$ and $W''(t) = (Z''(t) - 2\lambda Z'(t) + \lambda^2 Z(t))e^{-\lambda t}$. Substituting these derivatives into (4.5), we obtain

$$(P(Z''(t) - 2\lambda Z'(t) + \lambda^2 Z(t)), V) + \mathcal{B}^{(1)}(Z'(t) - \lambda Z(t), V) + \mathcal{B}^{(0)}(Z(t), V) = 0.$$

This relation can be rewritten as

$$(PZ''(t), V) + \mathcal{B}^{(1)}(Z'(t), V) - 2\lambda(PZ'(t), V) + \mathcal{B}^{(0)}(Z(t), V) - \lambda\mathcal{B}^{(1)}(Z(t), V) + \lambda^2(PZ(t), V) = 0. \tag{4.7}$$

In what follows, the following inequalities will play an essential role in the proof of Theorems 4.1 and 4.2. We will show that it is possible to choose a number $\lambda > 0$ so small that the following inequalities hold:

$$\mathcal{B}^{(1)}(V) - 2\lambda\|V\|^2_{[L_2(\Omega)]^6} \geq 0, \ \ \forall V \in \mathcal{K}_0, \tag{4.8}$$

$$\mathcal{B}^{(0)}(V) - \lambda\mathcal{B}^{(1)}(V) + \lambda^2\|V\|^2_{[L_2(\Omega)]^6} \geq \beta\|V\|^2_{[H^1(\Omega)]^6}, \ \ \beta > 0, \ \ \forall V \in \mathcal{K}_0. \tag{4.9}$$

Note that here and in what follows, for brevity, by $\mathcal{B}^{(q)}(V)$ we denote $\mathcal{B}^{(q)}(V, V)$, $q = 0, 1$.

Since $\operatorname{mes} S_1 > 0$, the bilinear forms $\mathcal{B}^{(q)}$, $q = 0, 1$ on the space $\mathcal{K}_0$ are strictly coercive, i.e., there exists a positive constant $\beta > 0$ such that for all $V \in \mathcal{K}_0$,

$$\mathcal{B}^{(q)}(V) \geq \beta\|V\|^2_{[H^1(\Omega)]^6}, \ \ q = 0, 1.$$

Let us choose $\lambda > 0$ small enough to satisfy the condition $\beta - 2\lambda > 0$. Then

$$\begin{aligned}\mathcal{B}^{(1)}(V) &\geq \beta\|V\|^2_{[H^1(\Omega)]^6} \\ &= (\beta - 2\lambda)\|V\|^2_{[H^1(\Omega)]^6} + 2\lambda\|V\|^2_{[H^1(\Omega)]^6} \geq (\beta - 2\lambda)\|V\|^2_{[H^1(\Omega)]^6} + 2\lambda\|V\|^2_{[L_2(\Omega)]^6}.\end{aligned}$$

From this relation we deduce

$$\mathcal{B}^{(1)}(V) - 2\lambda\|V\|^2_{[L_2(\Omega)]^6} \geq \beta_1\|V\|^2_{[H^1(\Omega)]^6} \ \ (\beta_1 = \beta - 2\lambda).$$

Since $\beta_1 > 0$, we obtain (4.8). Now we prove inequality (4.9). Note that the bilinear forms $\mathcal{B}^{(q)}$, $q = 0, 1$, satisfy the conditions

$$\mathcal{B}^{(q)}(V) \leq C\|V\|^2_{[H^1(\Omega)]^6}, \ \ \forall V \in [H^1(\Omega)]^6 \tag{4.10}$$

with some positive constant C. Since $\operatorname{mes} S_1 > 0$, there is a positive constant $\beta > 0$ such that

$$\mathcal{B}^{(0)}(V) \geq \beta\|V\|^2_{[H^1(\Omega)]^6}, \ \ \forall V \in \mathcal{K}_0.$$

Hence we have

$$\mathcal{B}^{(0)}(V)-\lambda\mathcal{B}^{(1)}(V)+\lambda^2\|V\|^2_{[L_2(\Omega)]^6}\geq\beta\|V\|^2_{[H^1(\Omega)]^6}-\lambda\mathcal{B}^{(1)}(V). \tag{4.11}$$

By virtue of (4.10), for $\lambda>0$, from (4.11) we obtain

$$\mathcal{B}^{(0)}(V)-\lambda\mathcal{B}^{(1)}(V)+\lambda^2\|V\|^2_{[L_2(\Omega)]^6}\geq(\beta-\lambda C)\|V\|^2_{[H^1(\Omega)]^6},\ \ \forall\,V\in\mathcal{K}_0.$$

Choose λ small enough such that $\beta-\lambda C\geq\frac{\beta}{2}$. Then it is clear that for such $\lambda>0$ inequality (4.9) holds.

Let us now prove Theorem 4.1. Substituting $Z'(t)\in\mathcal{K}_0$ for V in (4.7), we have

$$\begin{aligned}(PZ''(t),Z'(t))+\mathcal{B}^{(1)}(Z'(t))-2\lambda(PZ'(t),Z'(t))&\\ +\mathcal{B}^{(0)}(Z(t),Z'(t))-\lambda\mathcal{B}^{(1)}(Z(t),Z'(t))+\lambda^2(PZ(t),Z'(t))&=0,\end{aligned} \tag{4.12}$$

where the symbol $(\,\cdot\,,\,\cdot\,)$ denotes the scalar product of vector functions in the space $[L_2(\Omega)]^6$.

Let us write each term of (4.12) in a more detailed form as follows:

$$\begin{aligned}(PZ''(s),Z'(s))&=\int\limits_\Omega(PZ''(s),Z'(s))\,dx\\ &=\int\limits_\Omega\Big(\varrho\sum_{j=1}^3Z_j''(s)Z_j'(s)+J\sum_{j=4}^6Z_j''(s)Z_j'(s)\Big)\,dx\\ &=\frac{\varrho}{2}\sum_{j=1}^3\int\limits_\Omega\frac{d}{ds}\,(Z_j'(s))^2\,dx+\frac{J}{2}\sum_{j=4}^6\int\limits_\Omega\frac{d}{ds}\,(Z_j'(s))^2\,dx\\ &=\frac12\int\limits_\Omega\frac{d}{ds}\Big(\sum_{j=1}^3\big(\sqrt{\varrho}\,Z_j'(s)\big)^2+\sum_{j=4}^6\big(\sqrt{J}\,Z_j'(s)\big)^2\Big)\,dx,\\ (PZ'(s),Z'(s))&=\int\limits_\Omega\Big(\sum_{j=1}^3\big(\sqrt{\varrho}\,Z_j'(s)\big)^2+\sum_{j=4}^6\big(\sqrt{J}\,Z_j'(s)\big)^2\Big)\,dx,\\ \mathcal{B}^{(q)}(Z(s),Z'(s))&=\frac12\,\frac{d}{dx}\,\mathcal{B}^{(q)}(Z(s)),\\ (PZ(s),Z'(s))&=\frac12\int\limits_\Omega\frac{d}{ds}\Big(\sum_{j=1}^3\big(\sqrt{\varrho}\,Z_j(s)\big)^2+\sum_{j=4}^6\big(\sqrt{J}\,Z_j(s)\big)^2\Big)\,dx.\end{aligned}$$

Using these equalities, from (4.12) we get

$$\begin{aligned}&\frac12\int\limits_\Omega\frac{d}{ds}\Big(\sum_{j=1}^3\varrho\,(Z_j'(s))^2+\sum_{j=4}^6J(Z_j'(s))^2\Big)\,dx\\ &\quad+\frac12\bigg[\frac{d}{ds}\,\mathcal{B}^{(0)}(Z(s))+\lambda^2\int\limits_\Omega\frac{d}{ds}\Big\{\sum_{j=1}^3\varrho\,(Z_j'(s))^2+\sum_{j=4}^6J(Z_j'(s))^2\Big\}\,dx-\lambda\mathcal{B}^{(1)}(Z'(s))\bigg]\\ &\quad-2\lambda\int\limits_\Omega\Big\{\sum_{j=1}^3\varrho\,(Z_j'(s))^2+\sum_{j=4}^6J(Z_j'(s))^2\Big\}\,dx=0.\end{aligned} \tag{4.13}$$

Integrating (4.13) from zero to t, we have

$$\int\limits_{\Omega}\Big(\sum_{j=1}^{3}\varrho\,(Z_j'(t))^2+\sum_{j=4}^{6}J(Z_j'(t))^2\Big)\,dx+\mathcal{B}^{(0)}(Z(t))+\lambda^2\int\limits_{\Omega}\Big(\sum_{j=1}^{3}\varrho\,Z_j^2(t)+\sum_{j=4}^{6}JZ_j^2(t)\Big)\,dx$$
$$-\lambda\mathcal{B}^{(1)}(Z(t))+2\int\limits_0^t\Big\{\mathcal{B}^{(1)}(Z'(s))-2\lambda\int\limits_{\Omega}\Big[\sum_{j=1}^{3}\varrho\,(Z_j'(s))^2+\sum_{j=4}^{6}J(Z_j'(s))^2\Big]\,dx\Big\}\,ds$$
$$=\int\limits_{\Omega}\Big[\sum_{j=1}^{3}\varrho\,(Z_j'(0))^2+\sum_{j=4}^{6}J(Z_j'(0))^2\Big]\,dx+\mathcal{B}^{(0)}(Z(0))$$
$$+\lambda^2\int\limits_{\Omega}\Big[\sum_{j=1}^{3}\varrho\,(Z_j(0))^2+\sum_{j=4}^{6}J(Z_j(0))^2\Big]\,dx-\lambda\mathcal{B}^{(1)}(Z(0)). \quad (4.14)$$

Let $\alpha_0=\min\{\varrho,J,1\}$ and $\alpha_1=\max\{\varrho,J,1\}$. Then the right-hand side of equality (4.14) can be estimated by the quantity

$$\alpha_1\|Z'(0)\|^2_{[L_2(\Omega)]^6}+\mathcal{B}^{(0)}(Z(0))+\lambda^2\alpha_1\|Z(0)\|^2_{[L_2(\Omega)]^6}-\lambda\mathcal{B}^{(1)}(Z(0)), \quad (4.15)$$

whence it becomes clear that λ can be taken so small that (4.15) turns out to be a positive constant, which we denote by $\mathbb{C}$. From (4.14) we can write

$$\alpha_0\|Z'(t)\|_{[L^2(\Omega)]^2}+\mathcal{B}^{(0)}(Z(t))+\alpha_0\lambda^2\|Z(t)\|^2_{[L_2(\Omega)]^6}-\lambda\mathcal{B}^{(1)}(Z(t))$$
$$+2\int\limits_0^t\Big\{\mathcal{B}^{(1)}(Z'(s))-2\lambda\alpha_1\|Z'(s)\|^2_{[L_2(\Omega)]^6}\Big\}\,ds\le\mathbb{C}. \quad (4.16)$$

By virtue of formula (4.8), λ can be chosen in such a way that the integrand becomes non-negative and therefore we can discard this term. Again, λ can be taken so small that, by virtue of estimate (4.9) (after dividing both sides of (4.16) by α_0), we have

$$\|Z'(t)\|^2_{[L_2(\Omega)]^6}+\beta\|Z(t)\|^2_{[H^1(\Omega)]^6}\le\mathbb{C}_1. \quad (4.17)$$

We will show that inequality (4.2) and the first inequality of (4.3) follow from (4.17). Indeed, from (4.17) we obtain

$$\Big(\beta\|Z(t)\|^2_{[H^1(\Omega)]^6}\le\mathbb{C}_1\Big)\Longrightarrow\Big(\|Z(t)\|^2_{[H^1(\Omega)]^6}\le\frac{\mathbb{C}_1}{\beta}\Big)$$
$$\Longrightarrow\Big(\|e^{\lambda t}W(t)\|^2_{[H^1(\Omega)]^6}\le\mathbb{C}_2\Big)\Longrightarrow\Big(\|e^{\lambda t}(U(t)-\overline{U})\|^2_{[H^1(\Omega)]^6}\le\mathbb{C}_2\Big)$$
$$\Longrightarrow\Big(\|U(t)-\overline{U}\|^2_{[H^1(\Omega)]^6}\le\mathbb{C}_2e^{-2\lambda t}\Big)\Longrightarrow\Big(\|U(t)-\overline{U}\|_{[H^1(\Omega)]^6}\le\sqrt{\mathbb{C}_2}\,e^{-\gamma t}\Big)$$

for every $0<\gamma\le\lambda$.

Let us show the first inequality of (4.3). Further, from (4.17) we have

$$\sqrt{\mathbb{C}_1}\ge\|Z'(t)\|_{[L_2(\Omega)]^6}$$
$$=\|[U'(t)-\lambda(\overline{U}-U(t))]e^{\lambda t}\|_{[L_2(\Omega)]^6}\ge\Big\{\|U'(t)\|_{[L_2(\Omega)]^6}-\lambda\|\overline{U}-U(t)\|_{[L_2(\Omega)]^6}\Big\}e^{\lambda t}.$$

With the help of the relation

$$\|\overline{U}-U(t)\|_{[L_2(\Omega)]^6}\le\|\overline{U}-U(t)\|_{[H^1(\Omega)]^6}\le\sqrt{\mathbb{C}_2}\,e^{-\gamma t},\ \ \forall\,\gamma\in(0;\lambda],$$

we deduce

$$-\lambda\|\overline{U}-U(t)\|_{[L_2(\Omega)]^6}\ge-\lambda\sqrt{\mathbb{C}_2}\,e^{-\lambda t}\ge-\lambda\sqrt{\mathbb{C}_2}\,e^{-\gamma t}.$$

Therefore, from (4.16) we have

$$\sqrt{\mathbb{C}_1} \geq e^{\lambda t}\|U'(t)\|_{[L_2(\Omega)]^6} - \lambda\|\overline{U} - U(t)\|_{[L_2(\Omega)]^6} e^{\lambda t} \geq e^{\lambda t}\|U'(t)\|_{[L_2(\Omega)]^6} - \lambda\sqrt{\mathbb{C}_2}\,,$$

i.e., $e^{\lambda t}\|U'(t)\|_{[L_2(\Omega)]^6} \leq \mathbb{C}_3$, where $\mathbb{C}_3 = \sqrt{\mathbb{C}_1} + \lambda\sqrt{\mathbb{C}_2}$. Thus

$$\|U'(t)\|_{[L_2(\Omega)]^6} \leq \mathbb{C}_3 e^{-\lambda t} \leq \mathbb{C}_3 e^{-\gamma t}, \ \ \forall\,\gamma \in (0;\lambda],$$

and the first part of (4.3) is proved. Now, let us prove the second part of inequality (4.3).

By virtue of inequality (4.8), we can choose λ so small that the integrand in (4.14) will be non-negative, i.e., for $\beta_1 = \beta - 2\lambda\alpha_1$ we have

$$\mathcal{B}^{(1)}(V) - 2\lambda\|V\|^2_{[L_2(\Omega)]^6} \geq \beta_1\|V\|^2_{[H^1(\Omega)]^6}, \ \ \forall\, V \in \mathcal{K}_0.$$

Therefore, from (4.16), taking into account (4.17) we can conclude that

$$\|Z'(t)\|^2_{[L_2(\Omega)]^6} + \beta\|Z(t)\|^2_{[H^1(\Omega)]^6} + 2\beta_1\int\limits_0^t \|Z'(s)\|^2_{[H^1(\Omega)]^6}\,ds \leq \mathbb{C}.$$

Let $m = \min(1,\beta,2\beta_1)$. Then, by virtue of the previous inequality, we have

$$|Z'(t)|^2_{[L_2(\Omega)]^6} + \|Z(t)\|^2_{[H^1(\Omega)]^6} + \int\limits_0^t \|Z'(s)\|^2_{[H^1(\Omega)]^6}\,ds \leq \frac{\mathbb{C}}{m} = \widetilde{\mathbb{C}}. \tag{4.18}$$

This inequality is valid for all $t > 0$. Therefore,

$$\int\limits_0^\infty \|Z'(t)\|^2_{[H^1(\Omega)]^6}\,dt \leq \widetilde{\mathbb{C}}. \tag{4.19}$$

Note that $W = e^{-\lambda t}Z(t)$ and $W' = (Z'(t) - \lambda Z(t))e^{-\lambda t}$, i.e., $e^{\lambda t}W'(t) = Z'(t) - \lambda Z(t)$. Hence, for sufficiently small λ, we obtain

$$e^{-\gamma t}W'(t) = e^{(\gamma-\lambda)t}(Z'(t) - \lambda Z(t)), \ \ \forall\,\gamma < \lambda. \tag{4.20}$$

Since $W(t) = U(t) - \overline{U}$, we have $W'(t) = U'(t)$ and $e^{\gamma t}U'(t) = e^{\gamma t}W'(t)$. Let us show that $e^{\gamma t}W'(t) \in L_2(0,\infty;[H^1(\Omega)]^6)$. From (4.20) we have

$$\begin{aligned}
\int\limits_0^\infty \|e^{\gamma t}W'(t)\|^2_{[H^1(\Omega)]^6}\,dt &= \int\limits_0^\infty e^{2(\gamma-\lambda)t}\|Z'(t) - \lambda Z(t)\|^2_{[H^1(\Omega)]^6}\,dt \\
&\leq \int\limits_0^\infty e^{2(\gamma-\lambda)t}\big(\|Z'(t)\|_{[H^1(\Omega)]^6} + \lambda\|Z(t)\|_{[H^1(\Omega)]^6}\big)^2\,dt \\
&\leq 2\int\limits_0^\infty e^{2(\gamma-\lambda)t}\|Z'(t)\|^2_{[H^1(\Omega)]^6}\,dt + 2\lambda^2\int\limits_0^\infty e^{2(\gamma-\lambda)t}\|Z(t)\|^2_{[H^1(\Omega)]^6}\,dt.
\end{aligned} \tag{4.21}$$

Since $\gamma - \lambda < 0$, we have $e^{2(\gamma-\lambda)t} \leq 1$ on the interval $(0,\infty)$ and therefore the first term of (4.21), by virtue of (4.19), does not exceed the constant $2\widetilde{\mathbb{C}}$. From (4.18), $\forall\, t \in (0,\infty)$ $\|Z(t)\|^2_{[H^1(\Omega)]^6} \leq \widetilde{\mathbb{C}}$. Therefore, we have

$$\begin{aligned}
2\lambda^2\int\limits_0^\infty e^{2(\gamma-\lambda)t}\|Z(t)\|^2_{[H^1(\Omega)]^6}\,dt &= 2\lambda^2\lim_{A\to\infty}\int\limits_0^A e^{2(\gamma-\lambda)t}\|Z(t)\|^2_{[H^1(\Omega)]^6} \\
&\leq \widetilde{\mathbb{C}}\cdot 2\lambda^2\lim_{A\to\infty}\int\limits_0^A e^{2(\gamma-\lambda)t}\,dt = \mathbb{C}_1\lim_{A\to\infty}\frac{1}{2(\gamma-\lambda)}\,e^{2(\gamma-\lambda)t}\Big|_0^A \\
&= \widetilde{\mathbb{C}}_1\lim_{A\to\infty}(e^{2(\gamma-\lambda)A} - 1)\,\frac{1}{2(\gamma-\lambda)} = \widetilde{\mathbb{C}}_1\,\frac{1}{2(\gamma-\lambda)} = \widetilde{\mathbb{C}}_2,
\end{aligned}$$

which implies that $e^{\gamma t}U'(t) \in L_2(0,\infty;[H^1(\Omega)]^6)$. Theorem 4.1 is proved.

Proof of Theorem 4.2. Let $\widetilde{W}(t) = \widetilde{U}(t) - \overline{U}$, where $\widetilde{U}(t)$ is a solution to the quasi-static problem (3.11) and $\overline{U}$ is a solution to the static problem (4.1). From identities (3.11) and (4.1) it follows that $\widetilde{W}$ is a solution to the following variational equation:

$$\mathcal{B}^{(1)}(\widetilde{W}'(t), V) + \mathcal{B}^{(0)}(\widetilde{W}(t), V) = 0. \tag{4.22}$$

Indeed,

$$\begin{aligned}\mathcal{B}^{(1)}(\widetilde{W}'(t), V) + \mathcal{B}^{(0)}(\widetilde{W}(t), V) &= \mathcal{B}^{(1)}((\widetilde{U}(t) - \overline{U})', V)) + \mathcal{B}^{(0)}(\widetilde{U}(t) - \overline{U}, V) \\ &= \mathcal{B}^{(1)}(\widetilde{U}'(t), V) + \mathcal{B}^{(0)}(\widetilde{U}(t), V) - \mathcal{B}^{(0)}(\overline{U}, V) = (L, V) - (L, V) = 0.\end{aligned}$$

As in the previous case, substituting $e^{-\lambda t}Z(t)$ instead of $\widetilde{W}(t)$ in (4.22), we obtain

$$\begin{aligned}\mathcal{B}^{(1)}(e^{-\lambda t}(Z'(t) - \lambda Z(t)), V) + \mathcal{B}^{(0)}(e^{-\lambda t}Z(t), V) = 0,\\ e^{-\lambda t}\mathcal{B}^{(1)}(Z'(t), V) - \lambda e^{-\lambda t}\mathcal{B}^{(1)}(Z(t), V) + \mathcal{B}^{(0)}(Z(t), V)e^{-\lambda t} = 0.\end{aligned}$$

From this relation we have

$$\mathcal{B}^{(1)}(Z'(t), V) - \lambda\mathcal{B}^{(1)}(Z(t), V) + \mathcal{B}^{(0)}(Z(t), V) = 0. \tag{4.23}$$

We choose $\lambda > 0$ small enough such that the inequality

$$\mathcal{B}^{(0)}(V) - \lambda\mathcal{B}^{(1)}(V) \ge \beta_1\|V\|^2_{[H^1(\Omega)]^6}, \quad \forall V \in \mathcal{K}_0$$

holds for some $\beta_1 > 0$.

Indeed, in the previous case we have proved that it is possible to choose $\lambda > 0$ so small that inequality (4.9) holds implying the inequality

$$\mathcal{B}^{(0)}(V) - \lambda\mathcal{B}^{(1)}(V) \ge \beta\|V\|^2_{[H^1(\Omega)]^6} - \lambda^2\|V\|^2_{[L_2(\Omega)]^6} \ge (\beta - \lambda^2)\|V\|^2_{[H^1(\Omega)]^6},$$

where $\lambda > 0$ is a positive number such that $\lambda^2 < \beta$.

Further, let us substitute the function $Z'(t)$ into (4.23) instead of V (such a substitution is admissible, since $Z(t)$ and $Z'(t)$ belong to $\mathcal{K}_0$). We obtain

$$\mathcal{B}^{(1)}(Z'(s)) - \lambda\mathcal{B}^{(1)}(Z(s), Z'(s)) + \mathcal{B}^{(0)}(Z(s), Z'(s)) = 0,$$

whence

$$\int_0^t \mathcal{B}^{(1)}(Z'(s))\,ds + \frac{1}{2}\int_0^t \frac{d}{ds}\left(\mathcal{B}^{(0)}(Z(s) - \lambda\mathcal{B}^{(1)}(Z(s), Z'(s))\right)ds = 0,$$

$$2\beta\int_0^t \|Z'(s)\|^2_{[H^1(\Omega)]^6}\,ds + \mathcal{B}^{(0)}(Z(t)) - \lambda\mathcal{B}^{(1)}(Z(t)) \le 2\mathcal{B}^{(0)}(Z(0)) - 2\lambda\mathcal{B}^{(1)}(Z(0)),$$

$$2\beta\int_0^t \|Z'(s)\|^2_{[H^1(\Omega)]^6}\,ds + \beta_1\|Z(t)\|^2_{[H^1(\Omega)]^6} \le C \quad (\beta_1 = \beta - \lambda^2 > 0).$$

Since $2\beta > \beta_1$, we get

$$\|Z(t)\|^2_{[H^1(\Omega)]^6} + \int_0^t \|Z'(s)\|^2_{[H^1(\Omega)]^6}\,ds \le \mathbb{C} = \frac{C}{\beta_1}\,,$$

and, consequently,

$$\forall\, t > 0: \quad \|Z(t)\|^2_{[H^1(\Omega)]^6} \le \mathbb{C} \ \text{ and } \int_0^\infty \|Z'(s)\|^2_{[H^1(\Omega)]^6}\, ds \le \mathbb{C}. \tag{4.24}$$

From (4.24) it follows that

$$\mathbb{C} \ge \|Z(t)\|_{[H^1(\Omega)]^6} = \|e^{\lambda t}\widetilde{W}\|_{[H^1(\Omega)]^6} = e^{\lambda t}\|\widetilde{W}\|_{[H^1(\Omega)]^6} = e^{\lambda t}\|\widetilde{U}(t) - \overline{U}\|_{[H^1(\Omega)]^6}.$$

Therefore, $\|\widetilde{U}(t) - \overline{U}\|_{[H^1(\Omega)]^6} \le \mathbb{C}e^{-\lambda t}$. Let $\gamma > 0$ be an arbitrary positive number less than λ. Then we have $e^{\gamma t}\|\widetilde{U}(t) - \overline{U}\|_{[H^1(\Omega)]^6} \le \mathbb{C}e^{(\gamma-\lambda)t}$ and, since $\gamma - \lambda < 0$, we obtain $e^{(\gamma-\lambda)t} < 1$, $\forall\, t > 0$. Thus, we obtain the first inequality (4.4) of Theorem 4.2:

$$\|\widetilde{U}(t) - \overline{U}\|_{[H^1(\Omega)]^6} \le \mathbb{C}e^{-\gamma t}.$$

Since $\widetilde{W}'(t) = \widetilde{U}'(t)$, we have

$$e^{\gamma t}\widetilde{U}'(t) = e^{\gamma t}\widetilde{W}'(t) = e^{\gamma t}(e^{-\lambda t}Z(t))' = e^{(\gamma-\lambda)t}(Z'(t) - \lambda Z(t)).$$

Let us consider the integral

$$\int_0^\infty \|e^{\gamma t}\widetilde{U}'(t)\|^2_{[H^1(\Omega)]^6}\, dt.$$

Obviously,

$$\begin{aligned}
\int_0^\infty \|e^{\gamma t}\widetilde{U}'(t)\|^2_{[H^1(\Omega)]^6}\, dt &= \int_0^\infty e^{2(\gamma-\lambda)t}\|Z'(t) - \lambda Z(t)\|^2_{[H^1(\Omega)]^6}\, dt \\
&\le \int_0^\infty e^{2(\gamma-\lambda)t}\Big(\|Z'(t)\|_{[H^1(\Omega)]^6} + \lambda\|Z(t)\|_{[H^1(\Omega)]^6}\Big)^2\, dt \\
&\le 2\int_0^\infty e^{2(\gamma-\lambda)t}\|Z'(t)\|^2_{[H^1(\Omega)]^6}\, dt + 2\lambda^2\int_0^\infty e^{2(\gamma-\lambda)t}\|Z(t)\|^2_{1[H^1(\Omega)]^6}\, dt.
\end{aligned}$$

For $t \in (0, \infty)$ and $\gamma - \lambda < 0$, we have $e^{2(\gamma-\lambda)t} \le 1$. Thus, in view of the second inequality in (4.24), we get

$$2\int_0^\infty e^{2(\gamma-\lambda)t}\|Z'(t)\|^2_{[H^1(\Omega)]^6}\, dt \le 2\mathbb{C}.$$

From the first inequality in (4.24) we have

$$\begin{aligned}
&2\lambda^2\int_0^\infty e^{2(\gamma-\lambda)t}\|Z(t)\|^2_{[H^1(\Omega)]^6}\, dt \\
&\qquad \le 2\lambda^2\mathbb{C}\lim_{A\to\infty}\int_0^A e^{2(\gamma-\lambda)t}\, dt = \mathbb{C}_1\lim_{A\to\infty}\frac{1}{2(\gamma-\lambda)}\, e^{2(\gamma-\lambda)t}\Big|_0^A = \frac{\mathbb{C}_1}{2(\gamma-\lambda)}\,,
\end{aligned}$$

which proves that $e^{\gamma t}\widetilde{U}' \in L_2(0,\infty; V)$. Theorem 4.2 is proved. □

Now, let us show that from the above theorems Corollary 4.1 follows.

Under the conditions of Theorem 4.1, there exist positive numbers γ and C for which the following relations hold:

$$\|U(t)-\widetilde{U}(t)\|_{[H^1(\Omega)]^6} \le Ce^{-\gamma t}, \quad e^{\gamma t}(U'-\widetilde{U}') \in L_2(0,\infty;[H^1(\Omega)]^6),$$

where U and $\widetilde{U}$ are the solutions of the dynamic and quasi-static problems, respectively.

Indeed, if $\overline{U}$ is a solution of the static problem (4.1), then, by virtue of Theorems 4.1 and 4.2, we obtain

$$\begin{aligned}
&\|U(t)-\widetilde{U}(t)\|_{[H^1(\Omega)]^6} \\
&\quad = \|U(t)-\overline{U}+\overline{U}-\widetilde{U}(t)\|_{[H^1(\Omega)]^6} \le \|U(t)-\overline{U}\|_{[H^1(\Omega)]^6} + \|\overline{U}-\widetilde{U}(t)\|_{[H^1(\Omega)]^6} \le Ce^{-\gamma t}, \\
&\qquad e^{\gamma t}(U'(t)-\widetilde{U}'(t)) = e^{\gamma t}U'(t) - e^{\gamma t}\widetilde{U}'(t) \in L_2(0,\infty;[H^1(\Omega)]^6).
\end{aligned}$$

This completes Corollary 4.1.

5 Non-coercive case $S_1 = \varnothing$

Let $S_1 = \varnothing$. In this case, the quasi-static problem has solutions if and only if

$$(L,\varrho) = 0, \ \ \forall\, \varrho \in \mathcal{R}, \tag{5.1}$$

where $\mathcal{R}$ is the solution space to the equation $\mathcal{B}^{(q)}(V) = 0$, $q=0,1$, i.e., the space of rigid displacement vectors. This follows from (3.11) and from the relation

$$\mathcal{B}^{(q)}(V,\varrho) = 0, \ \ q=0,1, \ \ \forall\, V \in [H^1(\Omega)]^6.$$

Let $\dot{\mathcal{V}}$ denote the space of vectors, which is the quotient space of the space $\mathcal{V} = [H^1(\Omega)]^6$ with respect to $\mathcal{R}$, i.e., $\dot{\mathcal{V}} = \mathcal{V}/\mathcal{R}$. We denote the elements of this space by $\dot{V}$, which are the classes of equivalence. After the transition to the quotient space with respect to $\mathcal{R}$, the quasi-static problem will have a unique solution $\dot{\widetilde{U}}$ satisfying the following conditions:

$$\dot{\widetilde{U}} \in L_\infty(0,T;\dot{\mathcal{V}}), \quad \dot{\widetilde{U}}' \in L_2(0,T;\dot{\mathcal{V}}),$$
$$\mathcal{B}^{(1)}(\dot{\widetilde{U}}'(t),\dot{V}) + \mathcal{B}^{(0)}(\dot{\widetilde{U}}(t),\dot{V}) = (L,\dot{V}), \ \ \forall\, \dot{V} \in \dot{\mathcal{V}}.$$

Substituting $\varrho \in \mathcal{R}$ instead of V into (3.10) and taking into account the equality

$$\mathcal{B}^{(1)}(U'(t),\varrho) = \mathcal{B}^{(0)}(U(t),\varrho) = 0,$$

we obtain

$$(PU''(s),\varrho) = 0, \ \ \forall\, \varrho \in \mathcal{R}.$$

Integrating this equality from 0 to t and taking into account the initial conditions $U(0) = U'(0) = 0$, we get

$$\int\limits_0^t (PU''(s),\varrho)\,ds = \int\limits_0^t \frac{d}{ds}\,(PU'(s),\varrho)\,ds = (PU'(t),\varrho) = 0,$$
$$\int\limits_0^t (PU'(s),\varrho)\,ds = \int\limits_0^t \frac{d}{ds}\,(PU(s),\varrho)\,ds = (PU(t),\varrho) = 0, \ \ \forall\, \varrho \in \mathcal{R}.$$

These conditions are necessary for the existence of solutions to the dynamic frictionless problem.

Let us now identify U with the corresponding class $\dot{U}$, which is the solution to the following variational equation:

$$(P\dot{U}''(t),\dot{V}) + \mathcal{B}^{(1)}(\dot{U}'(t),\dot{V}) + \mathcal{B}^{(0)}(\dot{U}(t),\dot{V}) = (L,\dot{V}), \ \ \forall\, \dot{V} \in \dot{\mathcal{V}}.$$

Note that if U and V are arbitrary representatives of the classes $\dot{U}$ and $\dot{V}$, respectively, then, by definition, we get

$$\mathcal{B}^{(q)}(\dot{U},\dot{V})=\mathcal{B}^{(q)}(U,V),\ \ q=0,1.$$

Clearly, $\mathcal{B}^{(q)}(\dot{U},\dot{V})$ does not depend on the choice of the representatives U and V from the classes $\dot{U}$ and $\dot{V}$, respectively. Therefore, there is a positive number $\beta>0$ such that

$$\mathcal{B}^{(q)}(\dot{V})=\mathcal{B}^{(q)}(\dot{V},\dot{V})\geq\beta\|\dot{V}\|^2_{\dot{\mathcal{V}}},\ \ q=0,1, \tag{5.2}$$

where

$$\|\dot{V}\|_{\dot{\mathcal{V}}}=\inf_{\varrho\in\mathcal{R}}\|V+\varrho\|_{[H^1(\Omega)]^6}.$$

Let $\dot{\overline{U}}\in\dot{\mathcal{V}}$ be a solution to the following static equation:

$$\mathcal{B}^{(0)}(\dot{\overline{U}},\dot{V})=(L,\dot{V}),\ \ \forall\,\dot{V}\in\dot{\mathcal{V}}.$$

Inequality (5.2) implies the following assertion.

Theorem 5.1. *Let $S_1=\varnothing$. Suppose that all the conditions of Theorem 4.1 and condition (5.1) are satisfied. Then there exist the constants $\gamma>0$ and $C>0$ such that*

$$\|\dot{U}(t)-\dot{\overline{U}}\|_{\dot{\mathcal{V}}}\leq Ce^{-\gamma t},$$
$$|\dot{U}'(t)|_{\dot{\mathcal{H}}}\leq Ce^{-\gamma t},\ \ e^{\gamma t}\dot{U}'\in L_2(0,\infty;\dot{\mathcal{V}}),$$

where $H=[L_2(\Omega)]^6$ and $\dot{\mathcal{H}}=H/\mathcal{R}$.

Theorem 5.2. *Let the conditions of Theorem 5.1 be satisfied. Then there exist the constants $\gamma>0$ and $C>0$ such that*

$$\|\dot{\tilde{U}}'-\dot{\overline{U}}\|_{\dot{\mathcal{V}}}\leq Ce^{-\gamma t},\ \ e^{\gamma t}\dot{\tilde{U}}'\in L_2(0,\infty;\dot{\mathcal{V}}).$$

Clearly, these theorems imply the corollary similar to Corollary 4.1.

Corollary 5.1. *Under the conditions of Theorem 5.1, there exist the constants $\gamma>0$ and $C>0$ such that*

$$\|\dot{\tilde{U}}(t)-\dot{U}(t)\|_{\dot{\mathcal{V}}}\leq Ce^{-\gamma t},\ \ e^{\gamma t}(\dot{\tilde{U}}'-\dot{U}')\in L_2(0,\infty;\dot{\mathcal{V}}).$$

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