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WELL-POSEDNESS AND EXPONENTIAL STABILITY OF THE WAVE EQUATION WITH DELAY AND THERMODIFFUSION EFFECTS

Abstract. In this paper, we consider a wave equation with delay term and thermodiffusion effects. At first, we prove the existence and uniqueness of the system by the semigroup theory. Next, under appropriate assumptions, we prove the exponential stability of the solution by introducing a suitable Lyapunov functional.

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1 Introduction

Delay effects arise in many applications and practical problems because most phenomena, naturally, depend not only on the present state, but also on some past occurrences. We know that the dynamic systems with delay terms have become a major research subject in differential equation since the 1970s of the past century (see, e.g., [1, 2, 8, 9, 12–14]). In fact, in many cases it was shown that delay can be a source of instability and even an arbitrarily small delay may destabilize a system which is uniformly asymptotically stable in the absence of delay unless additional conditions or control terms have been used. For instance, in 1978, R. Datko [3] showed that the time delay in the velocity term can destabilize the system

$$\begin{cases} u_{tt}(x,t) = u_{xx}(x,t) - 2u_t(x,t-\tau) \ \text{ in } (0,1)\times(0,\infty), \\ u(0,t) = u(1,t) = 0, \ \ t \in (-\tau, +\infty), \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x) \ \text{ in } (0,1), \end{cases} \tag{1.1}$$

where $u = u(x,t)$ describes the displacement or rotational angle at spatial position x at time t.

The 1D wave equation is a second-order linear partial differential equation

$$u_{tt} - c^2 u_{xx} = 0, \tag{1.2}$$

where c denotes the speed of wave. This equation serves as an important mathematical model for the study of continuum dynamical systems. For example, longitudinal vibration of a beam [20], torsional vibration of a shaft and transverse vibration of a taut string [19] can be modeled by the 1D wave equation (1.2).

In 1986, R. Datko et al. [5] obtained the same result by replacing the internal delay in (1.1) by a time delay in the boundary feedback control. Then, in 1988, R. Datko [4] presented two examples of hyperbolic partial differential equations which are destabilized by small time delays in the boundary feedback controls.

In the n-dimensional case, it is well-known that the problem

$$\begin{cases} u_{tt}(x,t) - \Delta u(x,t) + \alpha_0 u_t(x,t) + \alpha u_t(x,t-\tau) = 0 \ \text{ in } \Omega\times(0,\infty), \\ u(x,t) = 0 \ \text{ on } \Gamma_0 \times (0,\infty), \\ \dfrac{\partial u}{\partial \nu}(x,t) = 0 \ \text{ on } \Gamma_1 \times (0,\infty), \end{cases} \tag{1.3}$$

is exponentially stable in the absence of delay ($\alpha = 0$, $\alpha_0 > 0$). In the presence of delay ($\alpha > 0$), in [15], S. Nicaise and C. Pignotti examined system (1.3) and proved that, under the assumption that the weight of the feedback is larger than the weight of the delay ($\alpha < \alpha_0$), the energy is exponentially stable. However, in the opposite case, they could produce a sequence of delays for which the corresponding solution is instable. S. A. Messaoudi et al. [9] considered a wave equation with a strong damping and a strong delay

$$\begin{cases} u_{tt}(x,t) - \Delta u(x,t) - \mu_1 \Delta u_t(x,t) - \mu_2 \Delta u_t(x,t-\tau) = 0 \ \text{ in } \Omega\times(0,\infty), \\ u(x,t) = 0 \ \text{ on } \partial\Omega \times (0,\infty), \\ u_t(x,t-\tau) = f_0(x,t-\tau), \ \ t\in(0,\tau), \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x) \ \text{ in } \Omega, \end{cases} \tag{1.4}$$

where Ω is a bounded and regular domain of $\mathbb{R}^n$, $\tau > 0$ represents the time delay, μ_1, μ_2 are real numbers such that $|\mu_2| < \mu_1$ and u_0, u_1, f_0 are the given data. The equation is regarded as a Kelvin–Voight linear model for a viscoelastic material in the presence of a delay response. In the second part of [9], the constant delay term in (1.4) is replaced by the distributed delay term of the form $-\int_{\tau_1}^{\tau_2} \mu_2(s)\Delta u_t(x,t-s)\,ds$, where $\mu_2 : [\tau_1,\tau_2] \to \mathbb{R}$ is a bounded function and $\tau_1 < \tau_2$ are two positive constants. They proved the well-posedness and established an exponential decay results under suitable conditions on the weights of the constant (respectively, distributed) delay and on the weight of damping terms.

On the other hand, it may not only destabilize a system which is asymptotically stable in the absence of delay, but may also lead to the ill posedness (see [4, 18] and the references therein). Therefore, the stability issue of systems with delay is of great theoretical and practical importance. In [18], R. Racke considered the following system with thermoelasticity:

$$\begin{cases} u_{tt}(x,t) - \alpha u_{xx}(x,t-\tau) + \gamma\theta_x(x,t) = 0, \\ \theta_t(x,t) - \kappa\theta_{xx}(x,t) + \gamma u_{xt}(x,t) = 0, \end{cases} \tag{1.5}$$

where α, γ, κ and L are some positive constants. The functions $u(x,t)$ and $\theta(x,t)$ describe, respectively, the displacement and the temperature difference, with $x \in (0, L)$ and $t \geq 0$. Moreover, $\tau > 0$ is the time delay. R. Racke proved that the internal time delay leads to ill-posedness of the system. However, the system without delay is exponentially stable (see, e.g., [7, 11, 17]). In [10], S. M. Khatir and F. Shel added to the delayed equation in system (1.5) a Kelvin–Voigt damping of the form $-\beta u_{xxt}(x,t)$ for some real positive number β which eventually depends on α, γ, κ and τ. They proved the well-posedness of the system by the semigroup theory. Next, under appropriate assumptions, they proved the exponential stability of the system by introducing a suitable Lyapunov functional.

In the present work, we introduce a wave equation model with delay, thermal, mass diffusion and thermoelastic effects. The equation is modeled by the following system:

$$\begin{cases} u_{tt} - bu_{xx} + \mu_1 u_t + \mu_2 u_t(x, t-\tau) - \zeta_1\theta_x - \zeta_2 C_x = 0, \\ \rho\theta_t + \varpi C_t - k\theta_{xx} - \zeta_1 u_{xt} = 0, \\ C_t - h(\zeta_2 u_x + \varrho C - \varpi\theta)_{xx} = 0, \end{cases} \tag{1.6}$$

where $(x,t) \in (0,1) \times (0, +\infty)$, $\tau > 0$ represents the time delay and μ_1, μ_2 are two positive constants. The function C denoted the concentration of the diffusive material in the elastic body. Here, $h > 0$ is the diffusion coefficient, ϖ is a measure of the thermodiffusion effect. In order to simplify the system, we use the following relation between chemical potential P and the concentration of the diffusion material C:

$$C = \frac{1}{\varrho}(P - \zeta_2 u_x + \varpi\theta).$$

Here, ϱ is a measure of the diffusive effect, we put

$$a = b - \frac{\zeta_2^2}{\varrho}, \quad \gamma_1 = \zeta_1 + \frac{\zeta_2\varpi}{\varrho}, \quad \gamma_2 = \frac{\zeta_2}{\varrho}, \quad c = \rho + \frac{\varpi^2}{\varrho}, \quad d = \frac{\varpi}{\varrho}, \quad r = \frac{1}{\varrho}.$$

Substituting in (1.6) the physical positive constants γ_1, γ_2, r, c and d satisfying

$$\lambda = rc - d^2 > 0, \tag{1.7}$$

the problem becomes

$$\begin{cases} u_{tt} - au_{xx} + \mu_1 u_t + \mu_2 u_t(x, t-\tau) - \gamma_1\theta_x - \gamma_2 P_x = 0, \\ c\theta_t + dP_t - k\theta_{xx} - \gamma_1 u_{xt} = 0, \\ d\theta_t + rP_t - hP_{xx} - \gamma_2 u_{xt} = 0, \end{cases} \tag{1.8}$$

where $(x,t) \in (0,1) \times (0, +\infty)$. This system is subjected to the boundary conditions

$$u(0,t) = u(1,t) = \theta(0,t) = \theta(1,t) = P(0,t) = P(1,t) = 0, \ \ \forall\, t \geq 0, \tag{1.9}$$

and the initial conditions

$$\begin{cases} u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \ \ x \in (0,1), \\ \theta(x,0) = \theta_0(x), \quad P(x,0) = P_0(x), \ \ x \in (0,1), \\ u_t(x, t-\tau) = f_0(x, t-\tau), \ \ (x,t) \in (0,1) \times (0,\tau). \end{cases} \tag{1.10}$$

The aim of this paper is to study the asymptotic stability of system (1.8)–(1.10) provided that (1.7) is satisfied. Here, we prove the well-posedness and stability results for problem (1.8)–(1.10) under the assumption

$$\mu_1 \geq |\mu_2|. \tag{1.11}$$

The main features of this paper are summarized as follows:

(a) In Section 2, we adopt the semigroup method to obtain the well-posedness of problem (1.8)–(1.10).

(b) In Section 3, we use the multiplier method to prove the exponential stability of problem (1.8)–(1.10).

2 Well-posedness

In this section, we give the existence and uniqueness result of problem (1.8)–(1.10) by using the semigroup theory. To this end, we first transform (1.8) into an equivalent problem by introducing, as in [15], a new dependent variable

$$z(x,\rho,t) = u_t(x, t-\rho\tau), \ \ x \in (0,1), \ \ \rho \in (0,1), \ \ t > 0.$$

Then we obtain

$$\tau z_t(x,\rho,t) + z_\rho(x,\rho,t) = 0, \ \ x \in (0,1), \ \ \rho \in (0,1), \ \ t > 0.$$

Hence system (1.8)–(1.10) is equivalent to

$$\begin{cases} u_{tt} - au_{xx} + \mu_1 u_t + \mu_2 z(x,1,t) - \gamma_1\theta_x - \gamma_2 P_x = 0, \ \ (x,t) \in (0,1)\times(0,+\infty), \\ c\theta_t + dP_t - k\theta_{xx} - \gamma_1 u_{xt} = 0, \ \ (x,t) \in (0,1)\times(0,+\infty), \\ d\theta_t + rP_t - hP_{xx} - \gamma_2 u_{xt} = 0, \ \ (x,t) \in (0,1)\times(0,+\infty), \\ \tau z_t(x,\rho,t) + z_\rho(x,\rho,t) = 0, \ \ (x,\rho,t) \in (0,1)\times(0,1)\times(0,\infty), \\ u(0,t) = u(1,t) = \theta(0,t) = \theta(1,t) = P(0,t) = P(1,t) = 0, \ \ \forall\, t \geq 0, \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \ \ x \in (0,1), \\ \theta(x,0) = \theta_0(x), \quad P(x,0) = P_0(x), \ \ x \in (0,1), \\ z(x,1,t) = f_0(x,t-\tau), \ \ t \in (0,\tau). \end{cases} \tag{2.1}$$

Introducing the vector function $U = (u, u_t, \theta, P, z)^T$, system (2.1) can be written as

$$\begin{cases} U'(t) = \mathcal{A}U(t), & t > 0, \\ U(0) = U_0 = (u_0, u_1, \theta_0, P_0, f_0)^T, \end{cases} \tag{2.2}$$

where the operator $\mathcal{A}$ is defined by

$$\mathcal{A}U = \begin{pmatrix} u_t \\ au_{xx} - \mu_1 u_t - \mu_2 z(x,1,t) + \gamma_1\theta_x + \gamma_2 P_x \\ \dfrac{rk}{\lambda}\,\theta_{xx} - \dfrac{hd}{\lambda}\,P_{xx} + \Big(\dfrac{r\gamma_1 - d\gamma_2}{\lambda}\Big)u_{tx} \\ \dfrac{ch}{\lambda}\,P_{xx} - \dfrac{kd}{\lambda}\,\theta_{xx} + \Big(\dfrac{c\gamma_2 - d\gamma_1}{\lambda}\Big)u_{tx} \\ -\dfrac{1}{\tau}\,z_\rho(x,\rho,t) \end{pmatrix}.$$

We introduce the following Hilbert space:

$$\mathcal{H} = H_0^1(0,1)\times L^2(0,1)\times L^2(0,1)\times L^2(0,1)\times L^2((0,1),L^2(0,1)).$$

For a positive constant ξ satisfying

$$\tau|\mu_2| < \xi < \tau\big(2\mu_1 - |\mu_2|\big), \tag{2.3}$$

we equip $\mathcal{H}$ with the inner product

$$(U,\widetilde{U})_{\mathcal{H}} = \int_0^1 u_t\widetilde{u}_t\,dx + \int_0^1 a u_x\widetilde{u}_x\,dx + \int_0^1 c\theta\widetilde{\theta}\,dx + \int_0^1 dP\widetilde{\theta}\,dx + \int_0^1 d\theta\widetilde{P}\,dx + \int_0^1 rP\widetilde{P}\,dx + \xi\int_0^1\int_0^1 z(x,\rho)\widetilde{z}(x,\rho)\,d\rho\,dx.$$

The domain of $\mathcal{A}$ is

$$D(\mathcal{A}) = \Big\{U\in\mathcal{H}:\ \ u\in H^2(0,1)\cap H_0^1(0,1),\ \ \theta,P\in H_0^1(0,1),\ \ z,z_\rho\in L^2((0,1),L^2(0,1)),\ \ z(x,0)=u_t(x)\Big\}.$$

Clearly, $D(\mathcal{A})$ is dense in $\mathcal{H}$.

We have the following existence and uniqueness result.

Theorem 2.1. *Under assumption* (1.11)*, for any* $U_0\in\mathcal{H}$*, there exists a unique weak solution* $U\in C(\mathbb{R}^+,\mathcal{H})$ *of problem* (2.2)*. Moreover, if* $U_0\in D(\mathcal{A})$*, then*

$$U\in C(\mathbb{R}^+,D(\mathcal{A}))\cap C^1(\mathbb{R}^+,\mathcal{H}).$$

Proof. To obtain the above result, we have to prove that $\mathcal{A}: D(\mathcal{A})\to\mathcal{H}$ is a maximal monotone operator. For this purpose, we need the following two steps: $\mathcal{A}$ is dissipative and $Id-\mathcal{A}$ is surjective.

Step 1. $\mathcal{A}$ is dissipative.

For any $U\in D(\mathcal{A})$, using the inner product and integration by parts, we can imply that

$$(\mathcal{A}U,U)_{\mathcal{H}} = -\mu_1\int_0^1 u_t^2\,dx - \mu_2\int_0^1 u_t z(x,1,t)\,dx - h\int_0^1 P_x^2\,dx - k\int_0^1\theta_x^2\,dx - \frac{\xi}{\tau}\int_0^1\int_0^1 z(x,\rho)z_\rho(x,\rho,t)\,d\rho\,dx. \tag{2.4}$$

Using Young's inequality, the second term in the right-hand side of (2.4) gives

$$-\mu_2\int_0^1 u_t z(x,1,t)\,dx \le \frac{|\mu_2|}{2}\int_0^1 u_t^2\,dx + \frac{|\mu_2|}{2}\int_0^1 z^2(x,1,t)\,dx.$$

Also, using integration by parts and the fact that $z(x,0,t)=u_t$, the last term in the right-hand side of (2.4) gives

$$\int_0^1\int_0^1 z(x,\rho,t)z_\rho(x,\rho,t)\,d\rho\,dx = \frac{1}{2}\int_0^1 z^2(x,1,t)\,dx - \frac{1}{2}\int_0^1 u_t^2\,dx.$$

Consequently, (2.4) yields

$$(\mathcal{A}U,U)_{\mathcal{H}} \le -\Big(\mu_1 - \frac{|\mu_2|}{2} - \frac{\xi}{2\tau}\Big)\int_0^1 u_t^2\,dx - h\int_0^1 P_x^2\,dx - k\int_0^1\theta_x^2\,dx - \Big(\frac{\xi}{2\tau} - \frac{|\mu_2|}{2}\Big)\int_0^1 z^2(x,1,t)\,dx,$$

and, using (2.3), we get

$$(\mathcal{A}U,U)_{\mathcal{H}} \leq -m_0\bigg(\int_0^1 u_t^2\,dx + \int_0^1 \theta_x^2\,dx + \int_0^1 P_x^2\,dx + \int_0^1 z^2(x,1,t)\,dx\bigg) \leq 0,$$

where

$$m_0 = \min\Big\{\mu_1 - \frac{|\mu_2|}{2} - \frac{\xi}{2\tau}, \frac{\xi}{2\tau} - \frac{|\mu_2|}{2}, h, k\Big\} \geq 0.$$

Hence the operator $\mathcal{A}$ is dissipative.

Step 2. $Id - \mathcal{A}$ is surjective.

To prove that the operator $Id-\mathcal{A}$ is surjective, we need to prove that for any $F = (f_1, f_2, f_3, f_4, f_5) \in \mathcal{H}$, there exists $U \in D(\mathcal{A})$ satisfying

$$(Id - \mathcal{A})U = F, \tag{2.5}$$

which is equivalent to

$$\begin{cases} u - u_t = f_1, \\ u_t - au_{xx} + \mu_1 u_t + \mu_2 z(x,1,t) - \gamma_1\theta_x - \gamma_2 P_x = f_2, \\ \lambda\theta - rk\theta_{xx} + hdP_{xx} - (r\gamma_1 - d\gamma_2)u_{tx} = \lambda f_3, \\ \lambda P - chP_{xx} + kd\theta_{xx} - (c\gamma_2 - d\gamma_1)u_{tx} = \lambda f_4, \\ \tau z(x,\rho,t) + z_\rho(x,\rho,t) = \tau f_5. \end{cases} \tag{2.6}$$

We note that the fifth equation in (2.6) with $z(x,0,t) = u_t(x,t)$ has a unique solution

$$z(x,\rho,t) = u(x)e^{-\tau\rho} - f_1(x)e^{-\tau\rho} + \tau e^{-\tau\rho}\int_0^\rho e^{\tau s} f_5(x,s)\,ds. \tag{2.7}$$

Clearly, $z, z_\rho \in L^2((0,1), L^2(0,1))$. Inserting $u_t = u - f_1$ and (2.7) in $(2.6)_2$, $(2.6)_3$ and $(2.6)_4$, we obtain

$$\begin{cases} \mu_0 u - au_{xx} - \gamma_1\theta_x - \gamma_2 P_x = g_1, \\ \lambda\theta - rk\theta_{xx} + hdP_{xx} - (r\gamma_1 - d\gamma_2)u_x = g_2, \\ \lambda P - chP_{xx} + kd\theta_{xx} - (c\gamma_2 - d\gamma_1)u_x = g_3, \end{cases} \tag{2.8}$$

where

$$\mu_0 = 1 + \mu_1 + \mu_2 e^{-\tau},$$

$$g_1 = \mu_0 f_1 + f_2 - \mu_2\tau e^{-\tau}\int_0^1 e^{\tau s} f_5(x,s)\,ds,$$

$$g_2 = \lambda f_3 - (r\gamma_1 - d\gamma_2)f_{1x}, \quad g_3 = \lambda f_4 - (c\gamma_2 - d\gamma_1)f_{1x}.$$

Multiplying $(2.8)_1$ by $\overline{u}$, $(2.8)_2$ by $\frac{c}{\lambda}\,\overline{\theta}$, $(2.8)_3$ by $\frac{r}{\lambda}\,\overline{P}$, $(2.8)_2$ by $\frac{d}{\lambda}\,\overline{P}$ and $(2.8)_3$ by $\frac{d}{\lambda}\,\overline{\theta}$ and integrating their sum over $(0,1)$, we can obtain the following variational equation

$$\mathcal{B}\big((u,\theta,P),(\overline{u},\overline{\theta},\overline{P})\big) = \mathcal{G}(\overline{u},\overline{\theta},\overline{P}), \tag{2.9}$$

where $\mathcal{B} : [H_0^1(0,1) \times L^2(0,1) \times L^2(0,1)]^2 \to \mathbb{R}$ is the bilinear form given by

$$\begin{aligned}\mathcal{B}\big((u,\theta,P),(\overline{u},\overline{\theta},\overline{P})\big) &= \mu_0\int_0^1 u\overline{u}\,dx + a\int_0^1 u_x\overline{u}_x\,dx + c\int_0^1 \theta\overline{\theta}\,dx + k\int_0^1 \theta_x\overline{\theta}_x\,dx + r\int_0^1 P\overline{P}\,dx \\ &+ h\int_0^1 P_x\overline{P}_x\,dx + d\int_0^1 (\theta\overline{P} + P\overline{\theta})\,dx + \gamma_2\int_0^1 (P\overline{u}_x - \overline{P}u_x)\,dx + \gamma_1\int_0^1 (\theta\overline{u}_x - \overline{\theta}u_x)\,dx,\end{aligned}$$

and $\mathcal{G} : [H_0^1(0,1) \times L^2(0,1) \times L^2(0,1)] \to \mathbb{R}$ is the linear form defined by

$$\mathcal{G}(\overline{u}, \overline{\theta}, \overline{P}) = \int_0^1 g_1 \overline{u}\, dx + \frac{c}{\lambda} \int_0^1 g_2 \overline{\theta}\, dx + \frac{r}{\lambda} \int_0^1 g_3 \overline{P}\, dx + \frac{d}{\lambda} \int_0^1 g_2 \overline{P}\, dx + \frac{d}{\lambda} \int_0^1 g_3 \overline{\theta}\, dx.$$

It is easy to verify that $\mathcal{B}$ is continuous and coercive, and $\mathcal{G}$ is continuous. Consequently, by the Lax-Milgram Lemma, system (2.9) has a unique solution

$$(u, \theta, P) \in H_0^1(0,1) \times L^2(0,1) \times L^2(0,1).$$

Applying the classical elliptic regularity, it follows from (2.9) that

$$(u, \theta, P) \in (H^2(0,1) \cap H_0^1(0,1)) \times H_0^1(0,1) \times H_0^1(0,1).$$

Hence there exists a unique $U \in D(\mathcal{B})$ such that (2.5) is satisfied. The operator $Id - \mathcal{A}$ is surjective. Consequently, the result of Theorem 2.1 follows from the Lumer–Phillips theorem (see [6, 16]). □

3 Exponential stability

In this section, we prove the exponential decay for system (2.2). It will be achieved by using the perturbed energy method. We define the energy functional $E(t)$ as

$$E(t) = \frac{1}{2} \int_0^1 \left[u_t^2 + a u_x^2 + c\theta^2 + 2dP\theta + rP^2 + \xi \int_0^1 z^2(x, \rho, t)\, d\rho \right] dx.$$

Noting (1.7), for $\theta, P \neq 0$ we have

$$c\theta^2 + 2d\theta P + rP^2 = \frac{\lambda}{r}\theta^2 + \left(\frac{d}{\sqrt{r}}\, \theta + \sqrt{r} P \right)^2 > 0,$$

whence we get that the energy $E(t)$ is positive.

The stability result reads as follows.

Theorem 3.1. *Let (u, θ, P, z) be a solution of (2.1) and assume that (1.11) holds. Then there exist two positive constants k_0 and k_1 such that*

$$E(t) \leq k_0 e^{-k_1 t}, \quad \forall\, t \geq 0.$$

The proof will be established through the following Lemmas.

Lemma 3.1. *Let (u, θ, P, z) be a solution of (2.2) and assume that (1.11) holds. Then we have the inequality*

$$E'(t) \leq -C_1 \int_0^1 u_t^2\, dx - k \int_0^1 \theta_x^2\, dx - h \int_0^1 P_x^2\, dx - C_2 \int_0^1 z^2(x, 1, t)\, dx \leq 0, \tag{3.1}$$

where

$$C_1 = \mu_1 - \frac{\xi}{2\tau} - \frac{|\mu_2|}{2}, \quad C_2 = \frac{\xi}{2\tau} - \frac{|\mu_2|}{2}.$$

Proof. Simple multiplication of equations $(2.1)_1$, $(2.1)_2$ and $(2.1)_3$ by u_t, θ and P, respectively, and integration over $(0,1)$, using integration by parts and the boundary conditions, yield

$$\frac{1}{2} \frac{d}{dt} \left\{ \int_0^1 u_t^2\, dx + \int_0^1 a u_x^2\, dx + \int_0^1 c\theta^2\, dx + \int_0^1 2dP\theta\, dx + \int_0^1 rP^2\, dx \right\}$$
$$= -\mu_1 \int_0^1 u_t^2\, dx - \mu_2 \int_0^1 u_t z(x, 1, t)\, dx - k \int_0^1 \theta_x^2\, dx - h \int_0^1 P_x^2\, dx. \tag{3.2}$$

Now, multiplying equation $(2.1)_4$ by $\frac{\xi}{\tau} z(x,\rho,t)$ and integrating over $(0,1)\times(0,1)$, and recalling that $z(x,0,t) = u_t(x,t)$, we obtain

$$\frac{\xi}{2}\frac{d}{dt}\int_0^1\int_0^1 z^2(x,\rho,t)\,d\rho\,dx = \frac{\xi}{2\tau}\int_0^1 u_t^2\,dx - \frac{\xi}{2\tau}\int_0^1 z^2(x,1,t)\,dx. \tag{3.3}$$

A combination of (3.2) and (3.3) gives

$$E'(t) = -\Big(\mu_1 - \frac{\xi}{2\tau}\Big)\int_0^1 u_t^2\,dx - \mu_2\int_0^1 u_t z(x,1,t)\,dx - k\int_0^1 \theta_x^2\,dx - h\int_0^1 P_x^2\,dx - \frac{\xi}{2\tau}\int_0^1 z^2(x,1,t)\,dx. \tag{3.4}$$

Meanwhile, using Young's inequality, we have

$$-\mu_2\int_0^1 u_t z(x,1,t)\,dx \le \frac{|\mu_2|}{2}\int_0^1 u_t^2\,dx + \frac{|\mu_2|}{2}\int_0^1 z^2(x,1,t)\,dx. \tag{3.5}$$

Simple substitution of (3.5) into (3.4) and use of (1.4) gives (3.1). The proof is complete. □

Lemma 3.2. *Let (u,θ,P,z) be a solution of (2.1). Then the functional*

$$L_1(t) = \int_0^1 u u_t\,dx,$$

satisfies the estimate

$$L_1'(t) \le -\frac{a}{2}\int_0^1 u_x^2\,dx + \Big(\frac{2\mu_1^2}{a}+1\Big)\int_0^1 u_t^2\,dx + \frac{2\gamma_1^2}{a}\int_0^1 \theta_x^2\,dx + \frac{2\mu_2^2}{a}\int_0^1 z^2(x,1,t)\,dx + \frac{2\gamma_2^2}{a}\int_0^1 P_x^2\,dx. \tag{3.6}$$

Proof. Taking the derivative of $L_1(t)$ with respect to t and using $(2.1)_1$, we have

$$L_1'(t) = -a\int_0^1 u_x^2\,dx + \int_0^1 u_t^2\,dx - \mu_1\int_0^1 u_t u\,dx - \mu_2\int_0^1 u z(x,1,t)\,dx + \gamma_1\int_0^1 u\theta_x\,dx + \gamma_2\int_0^1 uP_x\,dx. \tag{3.7}$$

Making use of Young's inequality and Poincaré's inequality, we obtain

$$-\mu_1\int_0^1 u_t u\,dx \le \frac{a}{8}\int_0^1 u_x^2\,dx + \frac{2\mu_1^2}{a}\int_0^1 u_t^2\,dx, \tag{3.8}$$

$$-\mu_2\int_0^1 u z(x,1,t)\,dx \le \frac{a}{8}\int_0^1 u_x^2\,dx + \frac{2\mu_2^2}{a}\int_0^1 z^2(x,1,t)\,dx, \tag{3.9}$$

$$\gamma_1\int_0^1 u\theta_x\,dx \le \frac{a}{8}\int_0^1 u_x^2\,dx + \frac{2\gamma_1^2}{a}\int_0^1 \theta_x^2\,dx, \tag{3.10}$$

$$\gamma_2\int_0^1 uP_x\,dx \le \frac{a}{8}\int_0^1 u_x^2\,dx + \frac{2\gamma_2^2}{a}\int_0^1 P_x^2\,dx. \tag{3.11}$$

Estimate (3.6) follows by substituting (3.8)–(3.11) into (3.7). □

Lemma 3.3. *Let (u, θ, P, z) be a solution of (2.1). Then the functions*

$$L_2(t) = \int_0^1 \int_0^1 e^{-2\tau\rho} z^2(x,\rho,t)\, d\rho\, dx$$

satisfies, for some positive constants n_1 and n_2, the estimates

$$L_2'(t) \le -n_1 \int_0^1 \int_0^1 z^2(x,\rho,t)\, d\rho\, dx - n_2 \int_0^1 z^2(x,1,t)\, dx + \frac{1}{\tau}\int_0^1 u_t^2\, dx. \tag{3.12}$$

Proof. Differentiating $L_2(t)$ with respect to t, and using equation $(2.1)_4$, we obtain

$$\begin{aligned} L_2'(t) &= -\frac{2}{\tau}\int_0^1 \int_0^1 e^{-2\tau\rho} z_\rho(x,\rho,t) z(x,\rho,t)\, d\rho\, dx \\ &= -2\int_0^1 \int_0^1 e^{-2\tau\rho} z^2(x,\rho,t)\, d\rho\, dx - \frac{1}{\tau}\int_0^1 \int_0^1 \frac{\partial}{\partial\rho}\left(e^{-2\tau\rho} z^2(x,\rho,t)\right) d\rho\, dx \\ &\le -m_0 \int_0^1 \int_0^1 z^2(x,\rho,t)\, d\rho\, dx - \frac{1}{\tau}\int_0^1 \int_0^1 \frac{\partial}{\partial\rho}\left(e^{-2\tau\rho} z^2(x,\rho,t)\right) d\rho\, dx. \end{aligned}$$

Simple integration of the last term, recalling that $z(x,0,t) = u_t$, gives the result. □

Now, we turn to prove our main result in this section.

Proof of Theorem 3.1. We define the Lyapunov functional $\mathcal{L}(t)$ by

$$\mathcal{L}(t) = NE(t) + L_1(t) + L_2(t),$$

where N is positive constant.

Differentiating $\mathcal{L}(t)$, exploiting (3.1), (3.6) and (3.12), we get

$$\begin{aligned} \mathcal{L}'(t) \le &- \left[C_1 N - \left(\frac{2\mu_1^2}{a} + 1\right) - \frac{1}{\tau}\right] \int_0^1 u_t^2\, dx - \frac{a}{2}\int_0^1 u_x^2\, dx - \left[kN - \frac{2\gamma_1^2}{a}\right]\int_0^1 \theta_x^2\, dx \\ &- \left[hN - \frac{2\gamma_2^2}{a}\right]\int_0^1 P_x^2\, dx - \left[C_2 N + n_2 - \frac{2\mu_2^2}{a}\right]\int_0^1 z^2(x,1,t)\, dx - n_1 \int_0^1 \int_0^1 z^2(x,\rho,t)\, d\rho\, dx. \end{aligned}$$

At this point, we choose N sufficiently large so that

$$N > \max\left\{\frac{1}{C_1}\left(\frac{2\mu_1^2}{a} + 1\right) + \frac{1}{\tau C_1}, \frac{2\gamma_1^2}{ak}, \frac{2\gamma_2^2}{ah}, \frac{2\mu_2^2}{aC_2} - \frac{n_2}{C_2}\right\}.$$

Consequently, from the above we deduce that there exist a positive constant α_0 such that

$$\mathcal{L}'(t) \le -\alpha_0 E(t). \tag{3.13}$$

On the other hand, it is not hard to see that $\mathcal{L}(t) \sim E(t)$, i.e., there exist two positive constants α_1 and α_2 such that

$$\alpha_1 E(t) \le \mathcal{L}(t) \le \alpha_2 E(t), \quad \forall\, t \ge 0. \tag{3.14}$$

A combination of (3.13) and (3.14) gives

$$\mathcal{L}'(t) \le -k_1 \mathcal{L}(t), \quad \forall\, t \ge 0, \tag{3.15}$$

where $k_1 = \frac{\alpha_0}{\alpha_2}$. A simple integration of (3.15) over $(0,t)$ yields

$$\mathcal{L}(t) \le \mathcal{L}(0) e^{-k_1 t}, \quad \forall\, t \ge 0.$$

Thus the conclusion of Theorem 3.1 follows. □

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