Memoirs on Differential Equations and Mathematical Physics

Volume ??, 2024, 1–11

Belkacem Chaouchi, Marko Kostić

PERIODIC SOLUTIONS FOR FOURTH-ORDER DIFFERENTIAL EVOLUTION EQUATION INVOLVING POLY-HARMONIC OPERATOR

Abstract. In this note, we analyze the existence and uniqueness of periodic solutions for a fourth-order evolution differential equation involving the well known poly-harmonic operator. The right-hand term of the equation is taken in some anistropic Hölder spaces. Our approach is based on the study of a fourth- order abstract differential equation. To this end, we opt for the use of analytic semigroup's techniques.

2020 Mathematics Subject Classification. 34G10, 34K10, 44A45, 47D06, 47G20.

Key words and phrases. abstract differential equations, Hölder spaces, periodic functions.

1 Motivation and statement of the problem

The theory of boundary value problems involving the poly-harmonic operator is a well-developed subject. For more information, we can refer the reader to [1–4, 17]. These works discuss a classical situation and establish the elliptic theory for the following model problem

$$\Delta^m u = f. \tag{1.1}$$

This differential equation is called the Kirchhoff–Love model for the vertical deflection of a thin elastic plate. The investigation of equation (1.1) was considered under different boundary conditions. The techniques of investigation are based on the use of the well-known potential theory or via the variational techniques. For the reader convenience, we just recall the classical poly-harmonic operator Δ^m which can be regarded as iterations of the Laplace operator, that is,

$$\Delta^m = \Delta(\Delta^{m-1}).$$

On the other hand, there exists another research axis which are concerned with the study of evolution equations involving a poly-harmonic operator. For instance, we quote two famous model equations

$$D_t u + (-\Delta)^m u = f,$$

and

$$D_t^2 u + (-\Delta)^m u = f.$$

These classes of problems have been well investigated in different contexts and under several conditions. For more details, we refer the readers to [6, 20] and the references therein. All these studies were motivated by the fact that this kind of problems arise in several models describing various phenomena in the applied sciences. In our situation, we deal with the solvability of Cauchy problems for the fourth-order evolution equations involving a relaxed poly-harmonic operator. More precisely, we provide a systematic study of the following equation:

$$\big(D_t + (-1)^m L_m\big)^4 u(t,x) = f(t,x), \ \ (t,x) \in \mathbb{R}^+ \times \Omega, \ \ m \in \mathbb{N} - \{0\}, \tag{1.2}$$

where L_m is the higher order differential operator defined by

$$L_m = \Big(\sum_{i=1}^{2} D_{x_i}^2 - \lambda\Big)^m, \ \ \lambda > 0,$$

Here, $x = (x_1, x_2)$ is a generic point of $\mathbb{R}^2$ and $\Omega = [0,1] \times [0,1]$. The right-hand term of equation (1.2) belongs to the anisotropic Hölder space $C^\theta(\mathbb{R}^+; L^p(\Omega))$ with $0 < \theta < 1$, $1 < p < +\infty$, consisting of all θ-Hölder continuous functions $f : \mathbb{R}^+ \to L^p(\Omega)$ such that

$$\|f\|_{C^\theta(\mathbb{R}^+;L^p(\Omega))} := \|f\|_{C_b(\mathbb{R}^+;L^p(\Omega))} + \sup_{t\in\mathbb{R}^+} \frac{\|f(t) - f(\tau)\|_{L^p(\Omega)}}{|t-\tau|^\theta} < \infty,$$

where

$$C_b(\mathbb{R}^+; L^p(\Omega)) = \Big\{ f \in C(\mathbb{R}^+; L^p(\Omega)) : \ \lim_{t\to+\infty} f(t,x) < +\infty \Big\}.$$

We also assume that f is a $T-$periodic function in the first variable

$$f(t+T, \cdot) = f(t, \cdot), \ \ t \in \mathbb{R}^+.$$

Let us supplement Eq. (1.2) with the following higher order boundary conditions:

$$\begin{cases} [L_m]^k \Big[\sum_{j=0}^{1} (-1)^{j+1} D_{x_1}^j \Big]^m u \Big|_{\mathbb{R}^+\times\{0\}\times[0,1]} = 0, \\ [L_m]^k u\big|_{\mathbb{R}^+\times\{1\}\times[0,1]} = 0, \end{cases} \tag{1.3}$$

and

$$[L_m]^k\Big[\sum_{j=0}^{2}D_{x_2}^{j}u\Big]^m\Big|_{\mathbb{R}^+\times[0,1]\times\{1\}}=0,\quad [L_m]^k\Big[\sum_{j=0}^{2}D_{x_2}^{j}u\Big]^m\Big|_{\mathbb{R}^+\times[0,1]\times\{0\}}=0, \tag{1.4}$$

where $k\in\{0,1,2,3\}$. We also impose the following initial conditions of periodic type:

$$D_t^j u\big|_{\{0\}\times\Omega}=D_t^j u\big|_{\{T\}\times\Omega},\;\; j=0,\dots,3. \tag{1.5}$$

The boundary conditions (1.4) involve the well known Wentzell type boundary conditions which were introduced in [22] in order to study a multidimensional Diffusion Processes. Note that in this work, the diversity of boundary and initial conditions imposed on equation (1.2) and the character of the functional framework make the use of classical approaches a difficult task. For these reasons, we will opt for the use of an abstract point of view. As in [13], the abstract version of problem (1.2)–(1.5) will be treated by using the theory of analytic semigroups. This allows us to obtain some interesting regularity results for our problem.

2 Operational formulation of problem (1.2)–(1.5)

We set $E=L^p(\Omega)$ endowed with its natural norm and define the vector-valued functions

$$\begin{aligned}u&:\mathbb{R}^+\to E;\;\; t\to u(t);\;\; u(t)(x)=u(t,x),\\ f&:\mathbb{R}^+\to E;\;\; t\to f(t);\;\; f(t)(x)=f(t,x).\end{aligned}$$

Consider the operator $\mathcal{A}_m$ defined by

$$\mathcal{A}_m\varphi(x)=((-1)^m L_m)\varphi(x) \tag{2.1}$$

and its natural domain

$$D(\mathcal{A}_m)=\Big\{\varphi,\;[L_m]\varphi\in E:\;\;\varphi\big|_{\{1\}\times[0,1]}=0\;\text{ and }\\ \Big(\sum_{j=0}^{1}(-1)^{j+1}D_{x_1}^{j}\Big)^m\varphi\Big|_{\{0\}\times[0,1]}=0,\;\;\Big(\sum_{j=0}^{2}D_{x_2}^{j}\Big)^m\varphi\Big|_{[0,1]\times\{0,1\}}=0\Big\}.$$

Then our problem (1.2) can be formulated as a complete fourth-order abstract differential equation

$$\begin{cases}u^{(4)}(t)+\displaystyle\sum_{k=1}^{3}\tbinom{4}{k}\mathcal{A}_m^k u^{(4-k)}(t)+\mathcal{A}_m^4u(t)=f(t),\quad t\in\mathbb{R}^+,\\ u^{(j)}(0)=u^{(j)}(T),\;\;1\le j\le 3,\;\;T>0,\end{cases} \tag{2.2}$$

with

$$\mathcal{A}_m^k=\mathcal{A}_m(\mathcal{A}_m^{k-1}).$$

To establish our main results, we need to use some spectral properties of the operator $\mathcal{A}_m$ defined by (2.1). It is obvious that this study needs to introduce the operator $(\mathcal{A}_0,D(\mathcal{A}_0))$ defined as follows:

$$\mathcal{A}_0\varphi(x)=\Big(\sum_{i=1}^{2}D_{x_i}^2-\lambda\Big)\varphi(x),\;\; x=(x_1,x_2)\in\Omega, \tag{2.3}$$

and its natural domain

$$D(\mathcal{A}_0)=\Big\{\varphi,\;\sum_{i=1}^{2}D_{x_i}^2\varphi\in E:\\ \varphi\big|_{\{0\}\times[0,1]}=0,\;\;\sum_{j=0}^{1}(-1)^{j+1}D_{x_1}^{j}\varphi\Big|_{\{1\}\times[0,1]}=0,\;\;\sum_{j=0}^{2}D_{x_2}^{j}\varphi\Big|_{[0,1]\times\{0,1\}}=0\Big\}.$$

For $f \in C^\theta(\mathbb{R}^+; L^p(\Omega))$, we search for a strict solution u of problem (2.2), that is, a vectorial function u such that

$$\begin{cases} \text{(i)} \quad u \in C_b^4(\mathbb{R}^+; E) \cap C_b(\mathbb{R}^+; D(\mathcal{A}_m^4)), \\ \text{(ii)} \quad u^{(4-k)} \in C_b(\mathbb{R}^+; D(\mathcal{A}_m^k)), \quad k = 1, 2, 3. \end{cases}$$

3 On the elliptic character of the operator $\mathcal{A}_m$

The investigation of spectral properties of operator (2.3) is based on the study of the following spectral problem:

$$\begin{cases} D_{x_1}^2 v(x_1, x_2) + D_{x_2}^2 v(x_1, x_2) - \lambda v(x_1, x_2) = \varphi(x_1, x_2), \quad (x_1, x_2) \in \Omega, \\ v\big|_{x_1=1} = 0, \ \sum\limits_{j=0}^{1} (-1)^{j+1} D_{x_1}^j v\big|_{x_1=0}, \ \sum\limits_{j=0}^{2} D_{x_2}^j v(\,\cdot\,, 0) = \sum\limits_{j=0}^{2} D_{x_2}^j v(\,\cdot\,, 1) = 0. \end{cases} \tag{3.1}$$

The study of (3.1) is performed in the Lebesgue space $L^p(\Omega)$. As in [7], we use the commutative version of the well-known sum's operator theory developed by G. da Prato and P. Grisvard in [15]. Briefly, the abstract version of (3.1) can be formulated as follows:

$$\mathcal{M}v + \mathcal{N}v - \lambda v = \varphi, \ \ v \in D(\mathcal{M}) \cap D(\mathcal{N}),$$

where $(\mathcal{M}, D(\mathcal{M}))$ and $(\mathcal{N}, D(\mathcal{N}))$ are the linear operators defined as

$$D(\mathcal{M}) = \Big\{ v, \ D_{x_1}^2 v \in L^p(0,1): \ \ D_{x_1} v\big|_{x_1=0} - v\big|_{x_1=0} = 0, \ \ v\big|_{x_1=1} = 0 \Big\}, \ \ \mathcal{M}v = D_{x_1}^2 v(x_1, x_2) \tag{3.2}$$

and

$$D(\mathcal{N}) = \Big\{ v, \ D_{x_2}^2 v \in L^p(0,1): \ \ \sum_{j=0}^{2} D_{x_2}^j v\Big|_{x_2=0} = \sum_{j=0}^{2} D_{x_2}^j v\Big|_{x_2=1} = 0 \Big\}, \ \ \mathcal{N}v = D_{x_2}^2 v(x_1, x_2).$$

The basic spectral properties of the operator $(\mathcal{M}, D(\mathcal{M}))$ are summarized in the following

Lemma 3.1. *Let $(\mathcal{M}, D(\mathcal{M}))$ be the linear operator defined by (3.2). Then $\mathcal{M}$ is a closed linear operator satisfying the following properties:*

(1) $\mathbb{R}^+ \supset \rho(\mathcal{M})$ *and* $\exists\, C > 0: \ \forall\, \mu \geq 0$,

$$\big\| (\mathcal{M} - \mu I)^{-1} \big\|_{L(E)} \leq \frac{C}{1 + \mu}\,, \tag{3.3}$$

($\rho(\mathcal{M})$ *is the resolvent set of* $\mathcal{M}$);

(2) $\overline{D(\mathcal{M})} = L^p(0,1)$.

Proof. As in [19, p. 89], an explicit calculus shows that

$$\big\{ (\mathcal{M} - \mu I)^{-1} \varphi \big\}(x_1) = -\int\limits_0^1 \mathcal{K}(\mu, s, x) \varphi(s)\, ds,$$

where $\varphi \in L^p(0,1)$ and

$$\mathcal{K}(\mu, s, x) = \begin{cases} \dfrac{\sinh \sqrt{\mu}\,(1 - x_1)[\sinh \sqrt{\mu}\, s + \sqrt{\mu}\, \cosh \sqrt{\mu}\, s]}{\sqrt{\mu}\,[\sinh \sqrt{\mu} + \sqrt{\mu}\, \cosh \sqrt{\mu}]}\,, & 0 \leqslant s \leqslant x_1, \\ \dfrac{\sinh \sqrt{\mu}\,(1 - s)[\sinh \sqrt{\mu}\, x_1 + \sqrt{\mu}\, \cosh \sqrt{\mu}\, x_1]}{\sqrt{\mu}\,[\sinh \sqrt{\mu} + \sqrt{\mu}\, \cosh \sqrt{\mu}]}\,, & x_1 \leqslant s \leqslant 1. \end{cases}$$

Here, $\sqrt{\mu}$ is the analytic determination defined by $\operatorname{Re}\sqrt{\mu} > 0$. Note that

$$\left|\sinh\sqrt{\mu} + \sqrt{\mu}\cosh\sqrt{\mu}\right| = \left|\frac{e^{\operatorname{Re}\sqrt{\mu}}}{2}\,(a_{\sqrt{\mu}} + ib_{\sqrt{\mu}}) + \frac{e^{-\operatorname{Re}\sqrt{\mu}}}{2}\,(c_{\sqrt{\mu}} + id_{\sqrt{\mu}})\right|,$$

where

$$\begin{cases} a_{\sqrt{\mu}} = 1 + \operatorname{Re}\sqrt{\mu}\,\cos\operatorname{Im}\sqrt{\mu} - \operatorname{Im}\sqrt{\mu}\,\sin\operatorname{Im}\sqrt{\mu}\,, \\ b_{\sqrt{\mu}} = 1 + \operatorname{Re}\sqrt{\mu}\,\sin\operatorname{Im}\sqrt{\mu} + \operatorname{Im}\sqrt{\mu}\,\cos\operatorname{Im}\sqrt{\mu}\,, \\ c_{\sqrt{\mu}} = (\operatorname{Re}\sqrt{\mu} - 1)\cos\operatorname{Im}\sqrt{\mu} - \operatorname{Im}\sqrt{\mu}\,\sin\operatorname{Im}\sqrt{\mu}\,, \\ d_{\sqrt{\mu}} = (1 - \operatorname{Re}\sqrt{\mu})\sin\operatorname{Im}\sqrt{\mu} + \operatorname{Im}\sqrt{\mu}\,\cos\operatorname{Im}\sqrt{\mu}\,. \end{cases}$$

So, we deduce that

$$\begin{aligned} &\left|\sinh\sqrt{\mu} + \sqrt{\mu}\cosh\sqrt{\mu}\right| \\ &\quad\geq \frac{e^{\operatorname{Re}\sqrt{\mu}}}{2}\left[\left(1 + \operatorname{Re}\sqrt{\mu}\right)^2 + \left(\operatorname{Im}\sqrt{\mu}\right)^2\right]^{1/2} - \frac{e^{-\operatorname{Re}\sqrt{\mu}}}{2}\left[\left(1 - \operatorname{Re}\sqrt{\mu}\right)^2 + \left(\operatorname{Im}\sqrt{\mu}\right)^2\right]^{1/2} \\ &\qquad\geq \sinh\operatorname{Re}\sqrt{\mu}\left[1 + \left(\operatorname{Re}\sqrt{\mu}\right)^2 + 2\operatorname{Re}\sqrt{\mu}\right]^{1/2}, \end{aligned}$$

which implies

$$\left|\sinh\sqrt{\mu} + \sqrt{\mu}\cosh\sqrt{\mu}\right| \geq \sinh\operatorname{Re}\sqrt{\mu}\left[1 + \left(\operatorname{Re}\sqrt{\mu}\right)\right].$$

To obtain the desired result, it suffices to observe that

$$\begin{aligned} \frac{\sinh(1-x_1)\operatorname{Re}\sqrt{\mu}}{\sinh\operatorname{Re}\sqrt{\mu}\,[1 + (\operatorname{Re}\sqrt{\mu})]} &= \frac{e^{\operatorname{Re}\sqrt{\mu}(1-x_1)} - e^{-\operatorname{Re}\sqrt{\mu}(1-x_1)}}{(e^{\operatorname{Re}\sqrt{\mu}} - e^{-\operatorname{Re}\sqrt{\mu}})(1 + (\operatorname{Re}\sqrt{\mu}))} \\ &= \frac{e^{-\operatorname{Re}\sqrt{\mu}x_1}\,[1 - e^{-\operatorname{Re}\sqrt{\mu}(2-x_1)}]}{(1 - e^{-2\operatorname{Re}\sqrt{\mu}})(1 + (\operatorname{Re}\sqrt{\mu}))} \leq \frac{Ce^{-\operatorname{Re}\sqrt{\mu}x_1}}{(1 + (\operatorname{Re}\sqrt{\mu}))}, \end{aligned}$$

from which it follows that

$$\int_0^1 \left|\frac{\sinh(1-x_1)\operatorname{Re}\sqrt{\mu}}{\sinh\operatorname{Re}\sqrt{\mu}\,[1 + (\operatorname{Re}\sqrt{\mu})]}\right|^p dx_1 \leq \int_0^1 |e^{-\operatorname{Re}\sqrt{\mu}x_1}|^p\,dx_1 \leq \frac{1}{p\operatorname{Re}\sqrt{\mu}}\,.$$

The density of $D(\mathcal{M})$ is obtained thanks to Proposition 1.1 in [18, p. 18]. □

Concerning the operator $(\mathcal{N}, D(\mathcal{N}))$, one has

Lemma 3.2. *Let* $(\mathcal{N}, D(\mathcal{N}))$ *be the linear operator defined by* (3.2). *Then* $\mathcal{N}$ *is a closed linear operator satisfying the following conditions:*

(1) $\mathbb{R}^+ \supset \rho(\mathcal{N})$ *and* $\exists\, C > 0 :\ \forall\,\mu \geq 0$,

$$\left\|(\mathcal{N} - \mu I)^{-1}\right\|_{L(E)} \leq \frac{C}{1+\mu}\,; \tag{3.4}$$

(2) $\overline{D(\mathcal{N})} = L^p(0,1)$.

Proof. These results use the same argument as in the previous lemma. In fact, using the classical variation of a constant, we show that

$$\left\{(\mathcal{N} - \mu I)^{-1}\varphi\right\}(x_2) = \frac{1}{(1 - e^{-2\sqrt{\mu}})}\int_0^1 e^{-\sqrt{\mu}(s - x_2 - 2)}\varphi(s)\,ds + \frac{1}{(1 - e^{-2\sqrt{\mu}})}\int_0^1 e^{-\sqrt{\mu}(2 - s + x_2)}\varphi(s)\,ds$$

$$+\frac{\mathcal{C}(\mu)}{(1-e^{-2\sqrt{\mu}})}\int_0^1 e^{-\sqrt{\mu}(2-s-x_2)}\varphi(s)\,ds+\frac{\mathcal{S}(\mu)}{(1-e^{-2\sqrt{\mu}})}\int_0^1 e^{-\sqrt{\mu}(s+x_2)}\varphi(s)\,ds$$

$$+\int_0^1 e^{-\sqrt{\mu}(s-x_2)}\varphi(s)\,ds+\frac{1}{2\sqrt{\mu}}\int_0^{x_2} e^{-\sqrt{\mu}(x_2-s)}\varphi(s)\,ds+\frac{1}{2\sqrt{\mu}}\int_0^{x_2} e^{-\sqrt{\mu}(s-x_2)}\varphi(s)\,ds,$$

where

$$\mathcal{C}(\mu)=\frac{(\mu-\sqrt{\mu}+1)}{(\mu+\sqrt{\mu}+1)}\quad\text{and}\quad\mathcal{S}(\mu)=\frac{1}{\mathcal{C}(\mu)}\,.$$

At this level, a direct computation shows that

$$\left\|(\mathcal{N}-\mu I)^{-1}\right\|_{L(E)}\leq\frac{C}{1+|\mu|}\,.$$

□

Now, we have

Proposition 3.1. *Let $(\mathcal{A}_0, D(\mathcal{A}_0))$ be the linear operator defined by* (2.3)*. Then $\mathcal{A}_0$ is a closed linear densely defined operator satisfying the natural ellipticity hypothesis*

$$\mathbb{R}^+\subset\rho(\mathcal{A}_0)\ \ \textit{and}\ \ \exists\, C>0,\ \ \forall\, z\geqslant 0\ \ \|(\mathcal{A}_0-zI)^{-1}\|\leqslant\frac{C}{1+z}\,. \tag{3.5}$$

Proof. As a direct consequence of the use of the commutative version of the sum operators technique due to [15], we conclude that the operator $(\mathcal{A}_0 - zI)^{-1}$ is well defined and

$$(\mathcal{A}_0-zI)^{-1}\varphi=-\frac{1}{2i\pi}\int_\Gamma(\mathcal{N}+zI)^{-1}(\mathcal{M}-zI-\lambda I)^{-1}\varphi\,dz, \tag{3.6}$$

where Γ is a suitable Jordan curve lying in $\sigma(-\mathcal{N})\cap\sigma(\mathcal{M})$. Keeping in mind estimates (3.3) and (3.4), estimate (3.5) can be easily deduced from formula (3.6). Furthermore, this estimate holds true if we replace z by $z+\lambda$. □

As a consequence of these results, we have

Lemma 3.3. *The operator $(\mathcal{A}_m, D(\mathcal{A}_m))$ defined by* (2.1) *is closed, densely defined and satisfies the Krein-ellipticity property, that is, $\mathbb{R}^+\subseteq\rho(\mathcal{A}_m)$ and there exists $C>0$ such that for all $z\geq 0$ we have*

$$\left\|(\mathcal{A}_m-zI)^{-1}\right\|_{\mathcal{L}(L^p(\Omega))}\leq\frac{C}{1+z}\,. \tag{3.7}$$

Proof. The proof is based on the use of the following algebraic formula:

$$(\mathcal{A}_m-\lambda I)^{-1}=\lambda^{1-\frac{1}{m}}\sum_{k=0}^{m-1}b_k\big((\mathcal{A}_0-\varepsilon_k\lambda^{\frac{1}{m}}I)^{-1}\big)$$

with

$$b_k=\frac{1}{\prod\limits_{n=0}^{m-1}(\varepsilon_k-\varepsilon_n)}\,,\ \ n\neq k,$$

where ε_k takes all the values of the mth root of unity. By adapting the same techniques used in [21], we conclude that for $\lambda>0$,

$$\left\|(\mathcal{A}_m-\lambda I)^{-1}\right\|_E=\mathcal{O}\Big(\frac{1}{\lambda}\Big).$$

□

Remark 3.1. Using the classical argument of continuation of the resolvent, we know that estimate (3.7) holds true in a sector of the form

$$\Sigma=\Big\{z\in\mathbb{C}^*:\ \ |\arg z|\leq\pi\Big(1-\frac{1}{2^m}\Big)\Big\}\cup B(0,\epsilon_0)$$

with some small $\epsilon_0>0$.

4 Construction of the solution of (2.2)

Differently from [9] and [10], we use another approach based essentially on the semigroup theory. In fact, from [5] we know that estimate (3.7) implies that the operator

$$\mathcal{B} = -(-\mathcal{A}_m)^{1/4}$$

is well defined and it is the infinitesimal generator of a generalized analytic semigroups $(e^{t\mathcal{B}})_{\xi>0}$. More precisely, there exists a sector

$$\Pi_{\delta,r_0} = \left\{ z \in \mathbb{C}^* : \ |\arg z| \leq \delta + \frac{\pi}{2} \right\} \cup \overline{B(0,r_0)}$$

(with some positive δ, r_0) and $C > 0$ such that

$$\rho(\mathcal{B}) \supset \Pi_{\delta,r_0} \ \text{ and } \ \forall\, z \in \Pi_{\delta,r_0}, \ \ \|(\mathcal{B} - zI)^{-1}\| \leq \frac{C}{1+|z|}\,.$$

Thus, for all $t > 0$ and $\varphi \in E$, one has

$$e^{t\mathcal{B}}\varphi = \frac{1}{2i\pi} \int\limits_{\gamma} e^{zt}(\mathcal{B} - zI)^{-1}\varphi\, dz,$$

where $\gamma = \partial\Pi_{\delta,r_0}$ (the sectorial boundary curve of Π_{δ,r_0} oriented from $\infty e^{i(\delta+\pi/2)}$ to $\infty e^{-i(\delta+\pi/2)}$). Then we have the following auxiliary results:

(1) $\forall\, k \in \mathbb{N}$, $\exists\, m_k \geq 1$, $\omega > 0$:
$$\|t^k \mathcal{B}^k e^{t\mathcal{B}}\|_{L(E)} \leq m_k e^{-\omega t}, \tag{4.1}$$

(2) $\lim\limits_{t\to 0} e^{t\mathcal{B}}\varphi = \varphi$ if and only if $\varphi \in \overline{D(\mathcal{B})}$.

Using the same reasoning as in [13], the representation formula of the solution for problem (2.2) can be deduced from the scalar case. Putting

$$\Psi(\mathcal{B}) = (1 - e^{T\mathcal{B}})^{-1}$$

and

$$\Phi(\mathcal{B}) = -\frac{1}{2}\,(1 + 4e^{\mathcal{B}T} + e^{2\mathcal{B}T})T^3,$$

the formal solution of (2.2) is formulated as follows:

$$u(t) = \sum_{k=1}^{4} u_k(t) \tag{4.2}$$

with

$$u_1(t) = \frac{\Psi(\mathcal{B})}{6} \int\limits_{t}^{t+T} (s-t)^3 e^{(t+T-s)\mathcal{B}} f(s)\, ds,$$

$$u_2(t) = -\frac{\Psi^2(\mathcal{B})}{2}\, T \int\limits_{t}^{t+T} (s-t)^2 e^{(t+T-s)\mathcal{B}} f(s)\, ds,$$

$$u_3(t) = \frac{\Psi^3(\mathcal{B})}{2}\, T^2 \int\limits_{t}^{t+T} (s-t) e^{(t+T-s)\mathcal{B}} f(s)\, ds,$$

$$u_4(t) = -\frac{\Phi(\mathcal{B})\Psi^4(\mathcal{B})}{2}\, T^3 \int\limits_{t}^{t+T} e^{(t+T-s)\mathcal{B}} f(s)\, ds.$$

Remark 4.1. For the sake of convenience, we note here that due to Lemma 3 in [21], the operator $(1-e^{T\mathcal{B}})^{-1}$ is well defined. In fact, it suffices to adapt the proof of [21, p. 59]. Note also that the absolute convergence of these integrals is justified by the key estimate (4.1).

First of all, we have

Proposition 4.1. *Let $f \in C^\theta(\mathbb{R}^+;E)$ with $\theta \in]0,1[$. Then, for all $t \geq 0$ and $k \in \mathbb{N}$,*

$$u(t) \in D(\mathcal{B}^k).$$

Proof. It suffices to show that

$$\|\mathcal{B}^k u(t)\|_{C^\theta(\mathbb{R}^+;E)} < \infty,$$

which means that for $i = 1,2,3,4$, we have

$$\|\mathcal{B}^k u_i(t)\|_{C^\theta(\mathbb{R}^+;E)} < \infty.$$

Since these vectorial functions can be treated similarly, we restrict ourselves on the first one. First, recall that

$$u_1(t) = \frac{e^{(t+T)\mathcal{B}}\Psi(\mathcal{B})}{6}\int_t^{t+T}(s-t)^3 e^{-\mathcal{B}s} f(s)\,ds,$$

since

$$e^{-t\mathcal{B}} \subset \bigcap_{k\geq 1} D(\mathcal{B}^k),$$

and the Banach valued functions Ψ and Φ are regular in the sense that there are bounded quantities. Then it follows that

$$\mathcal{B}^k u_1(t) = \frac{\mathcal{B}^k e^{(t+T)\mathcal{B}}}{6}\,\Psi(\mathcal{B})\bigg(\int_t^{t+T}(s-t)^3 e^{-\mathcal{B}s}(f(s)-f(t))\,ds - \int_t^{t+T}(s-t)^3\mathcal{B}^k e^{-\mathcal{B}s} f(t)\,ds\bigg),$$

and, clearly,

$$\begin{aligned}\|\mathcal{B}^k u_1(t)\|_E &\leqslant C\bigg(\int_0^{+\infty}(s-t)^{3+\theta}e^{-\omega s}\,ds\bigg)\|f\|_{C^\theta(\mathbb{R}^+;E)} + C'\|f(t)\|_E \\ &\leqslant C\Gamma(2+\theta)\|f\|_{C^\theta(\mathbb{R}^+;E)} + C'\|f(t)\|_E \leqslant C\|f\|_{C^\theta(\mathbb{R}^+;E)},\end{aligned}$$

where Γ is the usual Euler function defined by

$$\Gamma(z) = \int_0^{+\infty} e^{-w}w^{z-1}\,dw, \quad \operatorname{Re} z > 0. \qquad \square$$

The study of the regularity of solution (4.2) is purely technical. As for the techniques, we refer the reader to [8, 11–14, 16, 16] and the references therein, in which the kernel semigroups appearing in formula (4.2) were extensively used in different situation to provide some interesting results for different abstract differential equation under abstract boundary conditions.

Proposition 4.2. *Let $f \in C^\theta(\mathbb{R}^+;E)$ with $\theta \in]0,1[$. Then:*

(1) *for all $j \in \{1,2,3\}$: $u^{(j)} \in C_b(\mathbb{R}^+;E)$.*

(2) *for all $k \in \{0,1,2,3\}$: $\mathcal{A}_m^{4-k}u \in C_b(\mathbb{R}^+;E)$.*

Summing up, we are able to state our main regularity results for Problem (2.2).

Proposition 4.3. *Let $f \in C^{\theta}(\mathbb{R}^+; E)$ with $\theta \in]0,1[$. Then Problem* (2.2) *has a unique solution*

$$\begin{cases} \text{(i)} \quad u \in C_b^4(\mathbb{R}^+; E) \cap C_b(\mathbb{R}^+; D(A^4)), \\ \text{(ii)} \quad u^{(j)} \in C_b(\mathbb{R}^+; D(A^{4-j})), \quad j \in \{1,2,3\}. \end{cases}$$

Moreover, the solution u satisfies the following maximal regularity property

$$u^{(4)}, \mathcal{A}_m^j u^{(4-j)}, \mathcal{A}_m^4 u \in C^{\theta}(\mathbb{R}^+; E).$$

Now, we are able to give our main regularity results for our concrete problem (1.2)–(1.5) which can be formulated as follows.

Proposition 4.4. *Let $f \in C^{\theta}(\mathbb{R}^+; L^p(\Omega))$ with $0 < \theta < 1$, $1 < p < +\infty$. Then Problem* (1.2)–(1.5) *has a unique strict solution*

$$u \in C_b^4(\mathbb{R}^+; L^p(\Omega))$$

such that for all $(t,x) \in \mathbb{R}^+ \times \Omega$, we have

(1) $D_{x_1}^{2m} u(t,x)$, $D_{x_2}^{2m} u(t,x) \in C_b(\mathbb{R}^+; W^{8m,p}(\Omega))$;

(2) $D_t^4 u(t,x) \in C_b^4(\mathbb{R}^+; L^p(\Omega))$;

(3) $D_t^{4-k}[D_{x_1}^2 + D_{x_2}^2]^k u(t,x) \in C_b(\mathbb{R}^+; L^p(\Omega))$, $k \in \{1,2,3\}$.

Proof. First, the use of the sum operator technique allows us to conclude that

$$D_{x_1}^2 u(\,\cdot\,, x), D_{x_2}^2 u(\,\cdot\,, x) \in L^p(\Omega),$$

which justify the first assertion. The assertions (2) and (3) are viewed as direct consequence of the preceding theorem. □

References

[1] S. Agmon, On kernels, eigenvalues, and eigenfunctions of operators related to elliptic problems. *Comm. Pure Appl. Math.* **18** (1965), 627–663.

[2] M. S. Ashbaugh and R. D. Benguria, On Rayleigh's conjecture for the clamped plate and its generalization to three dimensions. *Duke Math. J.* **78** (1995), no. 1, 1–17.

[3] M. S. Ashbaugh, R. D. Benguria and R. S. Laugesen, Inequalities for the first eigenvalues of the clamped plate and buckling problems. *General inequalities, 7 (Oberwolfach, 1995)*, 95–110, Internat. Ser. Numer. Math., 123, *Birkhäuser, Basel*, 1997.

[4] M. S. Ashbaugh and R. S. Laugesen, Fundamental tones and buckling loads of clamped plates. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)* **23** (1996), no. 2, 383–402.

[5] A. V. Balakrishnan, Fractional powers of closed operators and the semigroups generated by them. *Pacific J. Math.* **10** (1960), 419–437.

[6] R. M. Brown and L. D. Gauthier, Inverse boundary value problems for polyharmonic operators with non-smooth coefficients. *Inverse Probl. Imaging* **16** (2022), no. 4, 943–966.

[7] B. Chaouchi, Solvability of second-order boundary-value problems on non-smooth cylindrical domains. *Electron. J. Differential Equations* **2013**, No. 199, 7 pp.

[8] B. Chaouchi, L. Benaissa and M. Kostić, On abstract third-order differential equation and its applications. *Acta Math. Sin. (Engl. Ser.)* **39** (2023), no. 3, 399–411.

[9] B. Chaouchi and M. Kostić, An efficient abstract method for the study of an initial boundary value problem on singular domain. Afr. Mat. 30 (2019), no. 3-4, 551–562.

[10] B. Chaouchi and M. Kostić, On the study of an initial value problem for a second order differential equation set in a singular cylindrical domain. *Sarajevo J. Math.* **15(28)** (2019), no. 1, 67–79.

[11] B. Chaouchi and M. Kostić, On solvability of a parabolic third-order differential equation set on singular domain. *Bull. Int. Math. Virtual Inst.* **11** (2021), no. 2, 223–236.

[12] B. Chaouchi and M. Kostić, On the study of the wave equation set on a singular cylindrical domain. *Publ. Inst. Math. (Beograd) (N.S.)* **112(126)** (2022), 13–22.

[13] B. Chaouchi, R. Labbas and B.-K. Sadallah, Laplace equation on a domain with a cuspidal point in little Hölder spaces. *Mediterr. J. Math.* **10** (2013), no. 1, 157–175.

205-224 (2004).

[14] M. Cheggag, A. Favini, R. Labbas, S. Maingot and A. Medeghri, Abstract differential equations of elliptic type with general Robin boundary conditions in Hölder spaces. *Appl. Anal.* **91** (2012), no. 8, 1453–1475.

[15] G. Da Prato and P. Grisvard, Sommes d'opárateurs linéaires et équations différentielles opérationnelles. (French) *J. Math. Pures Appl. (9)* **54** (1975), no. 3, 305–387.

[16] A. Favini, R. Labbas, H. Tanabe and A. Yagi, On the solvability of complete abstract differential equations of elliptic type. *Funkcial. Ekvac.* **47** (2004), no. 2, 205–224.

[17] F. Gazzola, H.-C. Grunau and G. Sweers, *Polyharmonic Boundary Value Problems. Positivity Preserving and Nonlinear Higher Order Elliptic Equations in Bounded Domains.* Lecture Notes in Mathematics, 1991. Springer-Verlag, Berlin, 2010.

[18] M. Haase, *The Functional Calculus for Sectorial Operators.* Operator Theory: Advances and Applications, 169. Birkhäuser Verlag, Basel, 2006.

[19] R. Haoua, Etude d'une équation différentielle abstraite avec conditions de Robin généralisées gouvernées par une classe d'opérateurs variables vérifiant l'hypothèse de Labbas–Terreni. *Thèse de doctorat, université de Mostaganem*, 2016.

[20] M. Karazym, T. Ozawa and D. Suragan, Multidimensional inverse Cauchy problems for evolution equations. *Inverse Probl. Sci. Eng.* **28** (2020), no. 11, 1582–1590.

[21] A. Lunardi, *Analytic Semigroups and Optimal Regularity in Parabolic Problems.* Modern Birkhäuser Classics. Birkhäuser/Springer Basel AG, Basel, 1995.

[22] A. D. Ventcel', On boundary conditions for multi-dimensional diffusion processes. *Theor. Probability Appl.* **4** (1959), 164–177.

(Received 30.12.2022; revised 01.08.2023; accepted 06.08.2023)

Authors' addresses:

Belkacem Chaouchi

Laboratory de l'Energie et des Systèmes Intelligents, Khemis Miliana University, 44225, Algeria.
E-mail: `chaouchicukm@gmail.com`

Marko Kostić

Faculty of Technical Sciences, University of Novi Sad, Trg D. Obradovića 6, 21125 Novi Sad, Serbia.
E-mail: `marco.s@verat.net`