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ON GENERALIZED CONFORMABLE FRACTIONAL CALCULUS ON TIME SCALES WITH APPLICATION TO A FRACTIONAL NONLOCAL THERMISTOR PROBLEM

Abstract. In this paper, we give a new general definition of conformable fractional derivative and integral on time scales, and study some of their important classical properties. As an application, the existence of solutions for the conformable fractional nonlocal thermistor problem on time scales is studied by using the Banach contraction principle and Schauder's fixed point theorem.

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1 Introduction

Fractional differential equations play an important role in describing many phenomena and processes in various fields of science such as physics, chemistry, control systems, population dynamics, etc. In 2014, a new fractional derivative, called the conformable fractional derivative, was introduced by Khalil et al. in [9]. Benkhettou et al. in [4] extended this definition to an arbitrary time scales. In [11], Sarikaya et al. introduced a new fractional derivative which generalizes the results obtained in [1] and [9].

Thermistor is a thermo-electric device constructed from a ceramic material whose electrical conductivity depends strongly on the temperature. Thermistors can be found in computers, digital thermostats, airplanes, portable heaters, cars, medical equipment, electrical outlets, chemical industries, etc. More details about the study of thermistor problems can be found, for example, in [2, 8, 12–17].

The existence of solutions was obtained by Bendouma et al. in [3] for the nonlocal nabla conformable fractional thermistor problem on time scale $\mathbb{T}$:

$$x_{\nabla}^{(\alpha)}(t) = \frac{\lambda f(t, x^{\rho}(t))}{\Big(\int\limits_a^b f(\tau, x^{\rho}(\tau))\,\nabla\tau\Big)^2} \text{ for all } t \in \mathbb{T}_{\kappa} \text{ and } x(b) = x_b \in \mathbb{R},$$

where λ is a fixed positive real, $f : \mathbb{T}_{\kappa} \times \mathbb{R}^+ \to \mathbb{R}_+^*$ is a continuous function and $x_{\nabla}^{(\alpha)}(t)$ denotes the nabla conformable fractional derivative of x at t of order $\alpha \in (0, 1)$.

This paper is organized as follows. In Section 2, we present some notations, definitions and results that are used throughout this paper. In Section 3 (resp. Section 4), we introduce a new fractional derivative (resp. fractional integral) which generalizes the results obtained in [4, 9, 11] and study its properties. In Section 5, as an application, we study the existence of solutions for the conformable fractional nonlocal thermistor problem on time scales by using Schauder's fixed point theorem and the Banach contraction principle.

2 Preliminaries

Let $\mathbb{T}$ be a time scale, which is a closed subset of $\mathbb{R}$. For $x \in \mathbb{T}$, we define the forward jump operator $\sigma : \mathbb{T} \to \mathbb{T}$ (resp. backward jump operator $\rho : \mathbb{T} \to \mathbb{T}$) by $\sigma(x) := \inf\{y \in \mathbb{T} : \ y > x\}$ (resp. $\rho(x) := \sup\{y \in \mathbb{T} : \ y < x\}$). We say that x is right-scattered (resp. left-scattered) if $\sigma(x) > x$ (resp. if $\rho(x) < x$); that x is isolated if it is right-scattered and left-scattered. The graininess function $\mu : \mathbb{T} \to [0, \infty)$ is defined by $\mu(x) := \sigma(x) - x$. If $\mathbb{T}$ has a left-scattered maximum M, then $\mathbb{T}^{\kappa} = \mathbb{T}\backslash\{M\}$, otherwise, $\mathbb{T}^{\kappa} = \mathbb{T}$. The backward graininess $\nu : \mathbb{T} \to [0, \infty)$ is defined by $\nu(x) := x - \rho(x)$. If $\mathbb{T}$ has a right-scattered minimum m, then $\mathbb{T}_{\kappa} = \mathbb{T} \setminus \{m\}$, otherwise, $\mathbb{T}_{\kappa} = \mathbb{T}$. For $a, b \in \mathbb{T}$, we define the closed interval $[a, b]_{\mathbb{T}} := \{x \in \mathbb{T} : \ a \le x \le b\}$.

$C([a, b]_{\mathbb{T}}, \mathbb{R})$ is the Banach space of all continuous functions from $[a, b]_{\mathbb{T}}$ into $\mathbb{R}$ with the norm

$$\|X\| = \sup\big\{|X(x)| : \ x \in [a, b]_{\mathbb{T}}\big\}.$$

Definition 2.1 ([5]). The function $f : \mathbb{T} \to \mathbb{R}$ is called rd-continuous provided it is continuous at right-dense point in $\mathbb{T}$ and has a left-sided limits at left-dense points in $\mathbb{T}$. We write $f \in C_{rd}(\mathbb{T}, \mathbb{R})$.

Definition 2.2 ([5]). For $f : \mathbb{T} \to \mathbb{R}$ and $x \in \mathbb{T}$, the delta derivative of f at x, denoted by $f^{\Delta}(x)$, is defined to be the number (provided it exists) with the property that given any $\varepsilon > 0$, there is a neighborhood U of x such that

$$\Big| f(\sigma(x)) - f(y) - f^{\Delta}(x)(\sigma(x) - y) \Big| \le \varepsilon |\sigma(x) - y| \text{ for all } y \in U.$$

Definition 2.3 ([4]). Let $f : \mathbb{T} \to \mathbb{R}$, $x \in \mathbb{T}^{\kappa}$ and $\alpha \in]0, 1]$. For $x > 0$, we define $f_{\Delta}^{(\alpha)}(x)$ to be the number (provided it exists) with the property that given any $\epsilon > 0$, there is a δ-neighborhood $\mathcal{V}_t \subset \mathbb{T}$ (i.e., $\mathcal{V}_x :=]x - \delta, x + \delta[\cap \mathbb{T})$ of x, $\delta > 0$, such that

$$\Big| \big[f(\sigma(x)) - f(y)\big] x^{1-\alpha} - f_{\Delta}^{(\alpha)}(x)[\sigma(x) - y] \Big| \le \epsilon |\sigma(x) - y| \text{ for all } x \in \mathcal{V}_x.$$

We call $f_{\Delta}^{(\alpha)}(x)$ the conformable fractional derivative of f of order α at t and define the conformable fractional derivative at 0 as $f_{\Delta}^{(\alpha)}(0) = \lim_{x\to 0^+} f_{\Delta}^{(\alpha)}(x)$.

Definition 2.4 ([9])**.** Given a function $f : [0,\infty) \to \mathbb{R}$ and a real constant $\alpha \in (0,1]$. The conformable fractional derivative of f of order α is defined by

$$f^{(\alpha)}(x) := \lim_{\varepsilon\to 0} \frac{f(x+\varepsilon x^{1-\alpha}) - f(x)}{\varepsilon}$$

for all $x > 0$.

If $f^{(\alpha)}(x)$ exists and is finite, we say that f is α-differentiable at x.

If f is α-differentiable in some interval $(0,a)$, $a > 0$, and $\lim_{x\to 0^+} f^{(\alpha)}(x)$ exists, then the conformable fractional derivative of f of order α at $x = 0$ is defined as

$$f^{(\alpha)}(0) = \lim_{x\to 0^+} f^{(\alpha)}(x).$$

Definition 2.5 ([11] (a-Conformable fractional derivative))**.** Given a function $f : [a,b] \to \mathbb{R}$ with $0 \le a < b$. Then the "a-conformable fractional derivative" of f of order $\alpha \in (0,1]$ is defined by

$$D_{\alpha}^{a}(f)(x) := \lim_{\varepsilon\to 0} \frac{f(x+\varepsilon x^{-\alpha}(x-a)) - f(x)}{\varepsilon(1-ax^{-\alpha})}$$

for all $x > a$.

Theorem 2.1 (Arzelà–Ascoli theorem [10])**.** *A subset $\mathcal{F}$ of $C([a,b],\mathbb{R}^n)$ is relatively compact (i.e., $\overline{\mathcal{F}}$ is compact) if and only if the following conditions hold:*

1. *$\mathcal{F}$ is uniformly bounded, i.e., there exists $M > 0$ such that*

$$\|f(x)\| < M \text{ for each } x \in [a,b] \text{ and each } f \in \mathcal{F}.$$

2. *$\mathcal{F}$ is equicontinuous, i.e., for every $\varepsilon > 0$, there exists $\delta > 0$ such that for each $x_1, x_2 \in [a,b]$, $|x_2 - x_1| \leqslant \delta$ implies $\|f(x_2) - f(x_1)\| \leqslant \varepsilon$, for every $f \in \mathcal{F}$.*

Theorem 2.2 (Schauder's fixed point theorem [7])**.** *Let C be a convex (not necessarily closed) subset of a normed linear space E. Then each compact map $N : C \to C$ has at least one fixed point.*

Theorem 2.3 (Banach's fixed point theorem [7])**.** *Let C be a non-empty closed subset of a Banach space E, then any contraction mapping N of C into itself has a unique fixed point.*

3 $\Delta_a^{(\alpha)}$-Conformable fractional derivative on time scales

In this section, we start with the definition which is a generalization of the conformable fractional derivative on time scales and study some of their important properties.

Definition 3.1. Let $f : \mathbb{T} \to \mathbb{R}$, $x \in \mathbb{T}^{\kappa}$ and $\alpha \in]0,1]$. For $x > a \ge 0$, $x^{\alpha} \ne a$, we define $\Delta_a^{(\alpha)}(f)(x)$ to be the number (provided it exists) with the property that given any $\epsilon > 0$, there is a δ-neighborhood $\mathcal{V}_x \subset \mathbb{T}$ (i.e., $\mathcal{V}_x :=]x-\delta, x+\delta[\cap \mathbb{T}$) of x, $\delta > 0$, such that

$$\left| \left[f(\sigma(x)) - f(y)\right]\phi_{\alpha}(x,a) - \Delta_a^{(\alpha)}(f)(x)[\sigma(x) - y] \right| \le \epsilon|\sigma(x) - y| \text{ for all } y \in \mathcal{V}_x,$$

where

$$\phi_{\alpha}(x,a) = \frac{x-a}{x^{\alpha}-a} = (x-a)(x^{\alpha}-a)^{-1}.$$

We call $\Delta_a^{(\alpha)}(f)(x)$ the $\Delta_a^{(\alpha)}$-conformable fractional derivative of f of order α at x and define the $\Delta_a^{(\alpha)}$-conformable fractional derivative at a as $\Delta_a^{(\alpha)}(f)(a) = \lim_{x\to a^+} \Delta_a^{(\alpha)}(f)(x)$.

Remark 3.1.

1. We have $\phi_0(x,a) = \frac{x-a}{1-a}$, $\phi_1(x,a) = 1$, $\phi_\alpha(x,0) = x^{1-\alpha}$.
2. If $\alpha = 1$, we have $\Delta_a^{(\alpha)}(f) = f^\Delta$ (see Definition 2.2).
3. If $a = 0$, then $\Delta_0^{(\alpha)}(f) = f_\Delta^{(\alpha)}$ is the delta-conformable fractional derivative of f of order α (see Definition 2.3).
4. If $\mathbb{T} = \mathbb{R}$ and $a = 0$, then $\Delta_0^{(\alpha)}(f) = f^{(\alpha)}$ is the conformable fractional derivative of f of order α (see Definition 2.4).
5. If $\mathbb{T} = \mathbb{R}$, then $\Delta_a^{(\alpha)}(f) = D_\alpha^a(f)$ is the a-conformable fractional derivative of f of order α (see Definition 2.5).

We denote:

(i) $$C^\alpha([a,b]_\mathbb{T},\mathbb{R}) = \Big\{f : [a,b]_\mathbb{T} \to \mathbb{R},\ f \text{ is the } \Delta_a^{(\alpha)}\text{-conformable fractional derivative of order } \alpha \text{ on } [a,b]_\mathbb{T} \text{ and } \Delta_a^{(\alpha)}(f) \in C([a,b]_\mathbb{T},\mathbb{R})\Big\}.$$

(ii) $$C_{rd}^\alpha([a,b]_\mathbb{T},\mathbb{R}) = \Big\{f : [a,b]_\mathbb{T} \to \mathbb{R},\ f \text{ is the } \Delta_a^{(\alpha)}\text{-conformable fractional derivative of order } \alpha \text{ on } [a,b]_\mathbb{T} \text{ and } \Delta_a^{(\alpha)}(f) \in C_{rd}([a,b]_\mathbb{T},\mathbb{R})\Big\}.$$

The following theorem is an analogue of Theorem 4 in [4] for the $\Delta_a^{(\alpha)}$-conformable fractional differentiable functions.

Theorem 3.1. *Let $\alpha \in]0,1]$ and $\mathbb{T}$ be a time scale. Assume $f : \mathbb{T} \to \mathbb{R}$ and let $x \in \mathbb{T}^\kappa$. The following properties hold:*

(i) *If f is $\Delta_a^{(\alpha)}$-conformal fractional differentiable of order α at $x > a$, then f is continuous at x.*

(ii) *If f is continuous at x and x is right-scattered, then f is $\Delta_a^{(\alpha)}$-conformable fractional differentiable of order α at x with*

$$\Delta_a^{(\alpha)}(f)(x) = \frac{f(\sigma(x)) - f(x)}{\mu(x)}\,\phi_\alpha(x,a) = \phi_\alpha(x,a) f^\Delta(x). \tag{3.1}$$

(iii) *If x is right-dense, then f is $\Delta_a^{(\alpha)}$-conformable fractional differentiable of order α at x if and only if the limit $\lim\limits_{y\to x} \frac{f(x)-f(y)}{(x-y)}\,\phi_\alpha(x,a)$ exists as a finite number. In this case,*

$$\Delta_a^{(\alpha)}(f)(x) = \phi_\alpha(x,a) f'(x). \tag{3.2}$$

(iv) *If f is $\Delta_a^{(\alpha)}$-conformable fractional differentiable of order α at x, then*

$$f(\sigma(x)) = f(x) + (\mu(x))\,\frac{1}{\phi_\alpha(x,a)}\,\Delta_a^{(\alpha)}(f)(x) = f(x) + (\mu(x))\,\frac{x^\alpha - a}{x-a}\,\Delta_a^{(\alpha)}(f)(x).$$

Proof.
(i). There exists a neighborhood $\mathcal{V}_x$ of x such that

$$\Big|(f(\sigma(x)) - f(y))\phi_\alpha(x,a) - \Delta_a^{(\alpha)}(f)(x)(\sigma(x)-y)\Big| \le \epsilon|\sigma(x)-y|$$

for $y \in \mathcal{V}_x$. Therefore,

$$
\begin{aligned}
|f(x)-f(y)| \leq & \left|(f(\sigma(x)-f(y))-\Delta_a^{(\alpha)}(f)(x)(\sigma(x)-y) \frac{1}{\phi_\alpha(x,a)}\right| \\
& + \left|(f(\sigma(x))-f(x))-\Delta_a^{(\alpha)}(f)(x)(\sigma(x)-x) \frac{1}{\phi_\alpha(x,a)}\right| \\
& + |\Delta_a^{(\alpha)}(f)(x)| \left|(\sigma(x)-y)-(\sigma(x)-x)\right| \left|\frac{1}{\phi_\alpha(x,a)}\right|
\end{aligned}
$$

for all $y \in \mathcal{V}_x \cap]x-\epsilon, x+\epsilon[$ and, since x is a right-dense point, we have

$$
\begin{aligned}
& |f(x)-f(y)| \\
& \leq \left|(f(\sigma(x)-f(y))-\Delta_a^{(\alpha)}(f)(x)(\sigma(x)-y) \frac{1}{\phi_\alpha(x,a)}\right| + \left|\Delta_a^{(\alpha)}(f)(x)(x-y)\right| \left|\frac{1}{\phi_\alpha(x,a)}\right| \\
& \leq \left(\epsilon\delta + |\Delta_a^{(\alpha)}(f)(x)|\delta\right) \left|\frac{1}{\phi_\alpha(x,a)}\right| \leq \delta\left(\epsilon + |\Delta_a^{(\alpha)}(f)(x)|\right).
\end{aligned}
$$

Since $\delta \to 0$ as $y \to x$, $x > 0$, and $|\frac{1}{\phi_\alpha(x,a)}| \leq 1$, this implies continuity of f at x.

(ii). By the continuity of f, we have

$$
\lim_{y \to x} \frac{f(\sigma(x))-f(y)}{\sigma(x)-y} \phi_\alpha(x,a) = \frac{f(\sigma(x))-f(x)}{\sigma(x)-x} \phi_\alpha(x,a) = \frac{f(x)-f(\sigma(x))}{\mu(x)} \phi_\alpha(x,a).
$$

Hence, given $\epsilon > 0$ and $\alpha \in]0,1]$, there is a neighborhood $\mathcal{V}_x$ of x such that

$$
\left|\frac{f(\sigma(x))-f(y)}{\sigma(x)-y} \phi_\alpha(x,a) - \frac{f(\sigma(x))-f(x)}{\mu(x)} \phi_\alpha(x,a)\right| \leq \epsilon
$$

for all $y \in \mathcal{V}_x$. It follows that

$$
\left|\,[f(\sigma(x))-f(y)]\phi_\alpha(x,a) - \frac{f(\sigma(x))-f(x)}{\mu(x)} \phi_\alpha(x,a)(\sigma(x)-y)\right| \leq \epsilon|\sigma(x)-y|
$$

for all $y \in \mathcal{V}_x$. The desired equality (3.1) follows from Definition 3.1.

(iii). Assume that f is $\Delta_a^{(\alpha)}$-conformable fractional differentiable of order α at x and x is right-dense. Let $\epsilon > 0$ be given. Since f is $\Delta_a^{(\alpha)}$-conformable fractional differentiable of order α at x, there is a neighborhood $\mathcal{V}_x$ of x such that $|[f(\sigma(x))-f(y)]\phi_\alpha(x,a) - \Delta_a^{(\alpha)}(f)(x)(\sigma(x)-y)| \leq \epsilon|\sigma(x)-y|$ for all $y \in \mathcal{V}_x$. Since $\sigma(x) = x$,

$$
\left|\frac{f(x)-f(y)}{x-y}\phi_\alpha(x,a) - \Delta_a^{(\alpha)}(f)(x)\right| \leq \epsilon
$$

for all $y \in \mathcal{V}_x$, $y \neq x$, and we get the desired result (3.2). Now, assume that the limit on the right-hand side of (3.2) exists and is equal to L, and x is left-dense. Then there exists $\mathcal{V}_x$ such that $|(f(x)-f(y))\phi_\alpha(x,a) - L(x-y)| \leq \epsilon|x-y|$ for all $y \in \mathcal{V}_x$. Since x is right-dense, we have

$$
\left|(f(\sigma(x))-f(y))\phi_\alpha(x,a) - L(\sigma(x)-y)\right| \leq \epsilon|\sigma(x)-y|,
$$

which leads to the conclusion that f is $\Delta_a^{(\alpha)}$-conformable fractional differentiable of order α at x and $\Delta_a^{(\alpha)}(f)(x) = L$.

(iv). If x is right-dense, i.e., $\sigma(x) = x$, then $\mu(x) = 0$ and

$$
f(\sigma(x)) = f(x) = f(x) - \mu(x)\Delta_a^{(\alpha)}(f)(x)\phi_\alpha(x,a).
$$

On the other hand, if x is right-scattered, i.e., $\sigma(x) > x$, then by (iii),

$$
f(\sigma(x)) = f(x) - \mu(x)\frac{1}{\phi_\alpha(x,a)} \cdot \frac{f(\sigma(x))-f(x)}{\mu(x)} \phi_\alpha(x,a) = f(x) - \mu(x) \frac{1}{\phi_\alpha(x,a)} \Delta_a^{(\alpha)}(f)(x).
$$

The proof is complete. □

Example 3.1.

(i) If $f : \mathbb{T} \to \mathbb{R}$ is defined by $f(x) = c$ for all $x \in \mathbb{T}$, $c \in \mathbb{R}$, then

$$\Delta_a^{(\alpha)}(f)(x) = \Delta_a^{(\alpha)}(c) = 0.$$

(ii) If $f : \mathbb{T} \to \mathbb{R}$ is defined by $f(x) = x$ for all $x \in \mathbb{T}$, then

$$\Delta_a^{(\alpha)}(f)(x) = \Delta_a^{(\alpha)}(x) = \begin{cases} \phi_\alpha(x,a) & \text{if } \alpha \neq 1, \\ 1 & \text{if } \alpha = 1. \end{cases}$$

(iii) Let $h > 0$ and $\mathbb{T} = h\mathbb{Z} := \{hk : k \in \mathbb{Z}\}$. Then $\sigma(x) = x + h$ and $\mu(x) = h$ for all $x \in \mathbb{T}$. For the function $f : x \in \mathbb{T} \mapsto x^2 \in \mathbb{R}$, we have

$$\Delta_a^{(\alpha)}(f)(x) = (2x + h)\phi_\alpha(x,a).$$

(iv) Let $p > 0$, fix $x_0 \in \mathbb{T}$ and $f(x) = e_p(x, x_0)$ for $x \in \mathbb{T}$. Then

$$\Delta_a^{(\alpha)}(f)(x) = p\phi_\alpha(x,a)e_p(x,x_0).$$

(v) Let $q > 1$ and $\mathbb{T} = q^{\mathbb{N}_0} := \{q^n : n \in \mathbb{N}_0\}$. For all $x \in \mathbb{T}$, we have $\sigma(x) = qx$ and $\mu(x) = (q-1)x$. Let $f : x \in \mathbb{T} \mapsto \log(x) \in \mathbb{R}$. Then

$$\Delta_a^{(\alpha)}(f)(x) = (\log(x))^{(\alpha)} = \frac{\log(q)}{(q-1)x}\,\phi_\alpha(x,a) = \frac{\log(q)}{(q-1)x}\,\frac{x-a}{x^\alpha - a}$$

for all $x \in \mathbb{T}$.

Example 3.2.

(i) The function $f : \mathbb{R} \to \mathbb{R}$ is $\Delta_a^{(\alpha)}$-conformable fractional differentiable of order α at the point $x \in \mathbb{R}$ if and only if the limit $\lim\limits_{y\to x} \frac{f(x)-f(y)}{x-y}\phi_\alpha(x,a)$ exists as a finite number. In this case,

$$\Delta_a^{(\alpha)}(f)(x) = \lim_{y\to x} \frac{f(x)-f(y)}{x-y}\,\phi_\alpha(x,a). \tag{3.3}$$

If $\alpha = 1$, then $\Delta_a^{(1)}(f) = f^\Delta(x) = f'(x)$. The identity (3.3) corresponds to the a-conformable derivative introduced in [11].

(ii) Let $h > 0$. If $f : h\mathbb{Z} \to \mathbb{R}$, then f is $\Delta_a^{(\alpha)}$-conformable fractional differentiable of order α at $x \in h\mathbb{Z}$ with

$$\Delta_a^{(\alpha)}(f)(x) = \frac{f(x+h) - f(x)}{h}\,\phi_\alpha(x,a).$$

If $\alpha = 1$ and $h = 1$, then $\Delta_a^{(1)}(f) = \Delta f(x) = f(x+1) - f(x)$, where Δ is the backward difference operator.

The following theorem is an analogue of Theorem 15 in [4] for the $\Delta_a^{(\alpha)}$-conformable fractional differentiable functions.

Theorem 3.2. *Assume $f, g : \mathbb{T} \to \mathbb{R}$ are $\Delta_a^{(\alpha)}$-conformable fractional differentiable of order α. Then:*

(i) *the sum $f + g$ is $\Delta_a^{(\alpha)}$-conformable fractional differentiable with*

$$\Delta_a^{(\alpha)}(f+g) = \Delta_a^{(\alpha)}(f) + \Delta_a^{(\alpha)}(g);$$

(ii) *for any $\lambda \in \mathbb{R}$, λf is $\Delta_a^{(\alpha)}$-conformable fractional differentiable with*

$$\Delta_a^{(\alpha)}(\lambda f) = \lambda \Delta_a^{(\alpha)}(f);$$

(iii) *if f and g are continuous, then the product fg is conformable fractional differentiable with*

$$\Delta_a^{(\alpha)}(fg) = \Delta_a^{(\alpha)}(f)g + (f\circ\sigma)\Delta_a^{(\alpha)}(g) = \Delta_a^{(\alpha)}(f)(g\circ\sigma) + f\Delta_a^{(\alpha)}(g);$$

(iv) *if f is continuous, then $1/f$ is $\Delta_a^{(\alpha)}$-conformable fractional differentiable with*

$$\Delta_a^{(\alpha)}\Big(\frac{1}{f}\Big) = -\frac{\Delta_a^{(\alpha)}(f)}{f(f\circ\sigma)},$$

valid at all points $x\in\mathbb{T}^\kappa$ for which $f(x)f(\sigma(x))\neq 0$;

(v) *if f and g are continuous, then f/g is $\Delta_a^{(\alpha)}$-conformable fractional differentiable with*

$$\Delta_a^{(\alpha)}\left(\frac{f}{g}\right) = \frac{\Delta_a^{(\alpha)}(f)g - f\Delta_a^{(\alpha)}(g)}{g(g\circ\sigma)},$$

valid at all points $x\in\mathbb{T}^\kappa$ for which $g(x)g(\sigma(x))\neq 0$.

Proof. Consider that $\alpha\in]0,1]$ and assume that f and g are $\Delta_a^{(\alpha)}$-conformable fractional differentiable at $x\in\mathbb{T}^\kappa$.
(i). Let $\epsilon>0$. Then there exist the neighborhoods $\mathcal{V}_x$ and $\mathcal{U}_x$ of x for which

$$\Big|\big[f(\sigma(x)) - f(y)\big]\phi_\alpha(x,a) - \Delta_a^{(\alpha)}(f)(x)(\sigma(x)-y)\Big| \leq \frac{\epsilon}{2}\,|\sigma(x)-y| \text{ for all } y\in\mathcal{V}_x$$

and

$$\Big|\big[g(\sigma(x)) - g(y)\big]\phi_\alpha(x,a) - \Delta_a^{(\alpha)}(g)(x)(\sigma(x)-y)\Big| \leq \frac{\epsilon}{2}\,|\sigma(x)-y| \text{ for all } y\in\mathcal{U}_x.$$

Let $\mathcal{W}_x = \mathcal{V}_x\cap\mathcal{U}_x$. Then

$$\Big|\big[(f+g)(\sigma(x)) - (f+g)(y)\big]\phi_\alpha(x,a) - \big[\Delta_a^{(\alpha)}(f)(x) + \Delta_a^{(\alpha)}(g)(x)\big](\sigma(x)-y)\Big| \leq \epsilon|\sigma(x)-y|$$

for all $y\in\mathcal{W}_x$. Thus $f+g$ is $\Delta_a^{(\alpha)}$-conformable differentiable at x and

$$\Delta_a^{(\alpha)}(f+g)(x) = \Delta_a^{(\alpha)}(f)(x) + \Delta_a^{(\alpha)}(g)(x).$$

(ii). Let $\epsilon>0$. Then

$$\Big|\big[f(\sigma(x)) - f(y)\big]\phi_\alpha(x,a) - \Delta_a^{(\alpha)}(f)(x)(\sigma(x)-y)\Big| \leq \epsilon|\sigma(x)-y|$$

for all y in a neighborhood $\mathcal{V}_x$ of x. It follows that

$$\Big|\big[(\lambda f)(\sigma(x)) - (\lambda f)(y)\big]\phi_\alpha(x,a) - \lambda\Delta_a^{(\alpha)}(f)(x)(\sigma(x)-y)\Big| \leq \epsilon|\lambda|\,|\sigma(x)-y| \text{ for all } y\in\mathcal{V}_x.$$

Therefore, λf is $\Delta_a^{(\alpha)}$-conformable fractional differentiable at x and $\Delta_a^{(\alpha)}(\lambda f) = \lambda\Delta_a^{(\alpha)}(f)$ holds at x.
(iii). If x is right-scattered, then

$$\begin{aligned}\Delta_a^{(\alpha)}(fg)(x) &= \Big[\frac{f(\sigma(x)) - f(x)}{\mu(x)}\,\phi_\alpha(x,a)\Big]g(\sigma(x)) + \Big[\frac{g(\sigma(x)) - g(x)}{\mu(x)}\,\phi_\alpha(x,a)\Big]f(x)\\ &= \Delta_a^{(\alpha)}(f)(x)g(\sigma(x)) + f(x)\Delta_a^{(\alpha)}(g)(x).\end{aligned}$$

If x is right-dense, then

$$\begin{aligned}\Delta_a^{(\alpha)}(fg)(x) &= \lim_{y\to x}\Big[\frac{f(x)-f(y)}{x-y}\,\phi_\alpha(x,a)\Big]g(x) + \lim_{y\to x}\Big[\frac{g(x)-g(y)}{x-y}\,\phi_\alpha(x,a)\Big]f(y)\\ &= \Delta_a^{(\alpha)}(f)(x)g(x) + \Delta_a^{(\alpha)}(g)(x)f(x) = \Delta_a^{(\alpha)}(f)(x)g(\sigma(x)) + \Delta_a^{(\alpha)}(g)(x)f(x).\end{aligned}$$

The other product rule formula follows by interchanging the role of functions f and g.

(iv). From Example 3.1 (i), we know that $\Delta_a^{(\alpha)}(f \cdot \frac{1}{f})(x) = \Delta_a^{(\alpha)}(1) = 0$. Therefore, by (iii),

$$\Delta_a^{(\alpha)}\Big(\frac{1}{f}\Big)(x) f(\sigma(x)) + \Delta_a^{(\alpha)}(f)(x)\,\frac{1}{f(x)} = 0.$$

Since we assume that $f(\sigma(x)) \neq 0$, we have

$$\Delta_a^{(\alpha)}\Big(\frac{1}{f}\Big)(x) = -\frac{\Delta_a^{(\alpha)}(f)(x)}{f(x)f(\sigma(x))}\,.$$

(v). We use (ii) and (iv) to obtain

$$\begin{aligned}\Delta_a^{(\alpha)}\Big(\frac{f}{g}\Big)(x) &= \Delta_a^{(\alpha)}\Big(f\cdot\frac{1}{g}\Big)(x) = f(x)\Delta_a^{(\alpha)}\Big(\frac{1}{g}\Big)(x) + \Delta_a^{(\alpha)}(f)(x)\,\frac{1}{g(\sigma(x))}\\ &= \frac{\Delta_a^{(\alpha)}(f)(x)g(x) - f(x)\Delta_a^{(\alpha)}(g)(x)}{g(x)g(\sigma(x))}\,.\end{aligned}$$

The proof is complete. □

Theorem 3.3 (Chain rule)**.** *Let $\alpha \in]0,1]$. Assume $g : \mathbb{T} \to \mathbb{R}$ is continuous and $\Delta_a^{(\alpha)}$-conformable fractional differentiable of order α at $x \in \mathbb{T}^\kappa$, and $f : \mathbb{R} \to \mathbb{R}$ is continuously differentiable. Then there exists c in the real interval $[x, \sigma(x)]$ with*

$$\Delta_a^{(\alpha)}(f \circ g)(x) = f'(g(c))\Delta_a^{(\alpha)}(g)(x).$$

Proof. The proof is done in a similar way as in [4]. □

Next, similar to Definition 30 in [17], we introduce the $\Delta_a^{(\alpha)}$-conformable fractional derivative on time scales for vector-valued functions.

Definition 3.2. Assume $f : \mathbb{T} \to \mathbb{R}^n$ is a function, $f(x) = (f_1(x), f_2(x), \ldots, f_n(x))$, $x \in \mathbb{T}^\kappa$, and $\alpha \in]0,1]$. For $x > a \geq 0$, $x^\alpha \neq a$, we define

$$\Delta_a^{(\alpha)}(f)(x) = \big(\Delta_a^{(\alpha)}(f_1)(x), \Delta_a^{(\alpha)}(f_2)(x), \ldots, \Delta_a^{(\alpha)}(f_n)(x)\big)$$

(provided it exists). $\Delta_a^{(\alpha)}(f)(x)$ is called the $\Delta_a^{(\alpha)}$-conformable fractional derivative of f of order α at x and we define the conformable fractional derivative at a as $\Delta_a^{(\alpha)}(f)(a) = \lim_{x\to a^+} \Delta_a^{(\alpha)}(f)(x)$.

We define the $\Delta_a^{(\alpha)}$-conformable fractional differentiable of f of order $\alpha \in (m, m+1]$, where m is some natural number.

Definition 3.3. Let $\mathbb{T}$ be a time scale, $\alpha \in (m, m+1]$, $m \in \mathbb{N}$, and let f be m times delta differentiable at $x \in \mathbb{T}_{\kappa^m}$. We define the $\Delta_a^{(\alpha)}$-conformable fractional differentiable of f of order α as $\Delta_a^{(\alpha)}(f)(x) = \Delta_a^{(\alpha-m)}(f^{\Delta^m})(x)$.

4 Delta (α_a)-conformable fractional integral

Now, we introduce the $\Delta_{(\alpha,a)}$ conformable fractional integral (or delta (α_a)-fractional integral) on time scales.

Definition 4.1. Let $a, b \in \mathbb{T}$ with $a < b$ and $f : \mathbb{T} \to \mathbb{R}$ be a regulated function. Then the delta (α_a)-fractional integral of f, $0 < \alpha \leq 1$, is defined by

$$\int f(x)\,\Delta_\alpha^a x = \int f(x)(\phi_\alpha(x,a))^{-1}\,\Delta x = \int \frac{x^\alpha - a}{x-a} f(x)\,\Delta x.$$

Note that if $\alpha = 1$, then $\int f(x)\,\Delta_\alpha^a x = \int f(x)\,\Delta x$ is the integral given in [6]. If $\mathbb{T} = \mathbb{R}$, then $\int f(x)\,\Delta_\alpha^a x = \int f(x)\,\frac{x^\alpha - a}{x-a}\,\Delta x$ is the $(\alpha; a)$-conformable fractional integral given in [11].

Definition 4.2. Assume $f : \mathbb{T} \to \mathbb{R}$ is a function. Let A be a Δ-measurable subset of $\mathbb{T}$. Then f is delta (α_a)-integrable on A if and only if $\frac{x^\alpha - a}{x-a} f(x)$ is integrable on A and $\int\limits_A f(x)\,\Delta_\alpha^a x = \int\limits_A \frac{x^\alpha-a}{x-a} f(x)\,\Delta x$. Then for $A = [a,x]_{\mathbb{T}}$, we have

$$\int\limits_{[a,x]_{\mathbb{T}}} f(y)\,\Delta_\alpha^a y = \int\limits_a^x f(y)\,\Delta_\alpha^a y = \int\limits_a^x \frac{y^\alpha - a}{y-a} f(y)\,\Delta y.$$

Definition 4.3. Suppose $f : \mathbb{T} \to \mathbb{R}$ is a regulated function. Denote the indefinite delta (α_a)-fractional integral of f of order α, $\alpha \in (0,1]$, as follows: $F_{\Delta_{\alpha,a}}(x) = \int f(x)\,\Delta_\alpha^a x$. Then for all $a_1, b_1 \in \mathbb{T}$, we define the Cauchy delta (α_a)-fractional integral by

$$\int\limits_{a_1}^{b_1} f(x)\,\Delta_\alpha^a x = F_{\Delta_{\alpha,a}}(b_1) - F_{\Delta_{\alpha,a}}(a_1).$$

Theorem 4.1. *Let $\alpha \in (0,1]$. Then for any rd-continuous function $f : \mathbb{T} \to \mathbb{R}$, there exists a function $F_{\Delta_{(\alpha,a)}} : \mathbb{T} \to \mathbb{R}$ such that $\Delta_a^{(\alpha)}(F_{\Delta_{\alpha,a}})(x) = f(x)$ for all $x \in \mathbb{T}^\kappa$. The function $F_{\Delta_{(\alpha,a)}}$ is said to be a delta (α_a)-antiderivative of f.*

Proof. The case $\alpha = 1$ is proved in [5]. Let $\alpha \in (0,1)$. Suppose f is rd-continuous. By Theorem 1.16 of [6], f is regulated. Then $F_{\Delta_{(\alpha,a)}}(x) = \int f(x)\,\Delta_\alpha^a x$ is $\Delta_a^{(\alpha)}$-conformable fractional differentiable on $\mathbb{T}^\kappa$. Using Definition 4.1, we obtain

$$\Delta_a^{(\alpha)}(F_{\Delta_{(\alpha,a)}})(x) = \frac{x-a}{x^\alpha - a}\left(F_{\Delta_{(\alpha,a)}}(x)\right)^\Delta = f(x), \quad x \in \mathbb{T}^\kappa. \qquad \square$$

Similar to Theorem 31 in [4], we present the following theorem.

Theorem 4.2. *Let $\alpha \in (0,1]$, $\tau, b, c \in \mathbb{T}$, $\lambda, \gamma \in \mathbb{R}$, and let f, g be two rd-continuous functions. Then:*

(i) $$\int\limits_\tau^b \left[\lambda f(x) + \gamma g(x)\right]\Delta_\alpha^a x = \lambda \int\limits_\tau^b f(x)\,\Delta_\alpha^a x + \gamma \int\limits_\tau^b g(x)\,\Delta_\alpha^a x;$$

(ii) $$\int\limits_\tau^b f(x)\,\Delta_\alpha^a x = -\int\limits_b^\tau f(x)\,\Delta_\alpha^a x;$$

(iii) $$\int\limits_\tau^b f(x)\,\Delta_\alpha^a x = \int\limits_\tau^c f(x)\,\Delta_\alpha^a x + \int\limits_c^b f(x)\,\Delta_\alpha^a x;$$

(iv) $$\int\limits_\tau^a f(x)\,\Delta_\alpha^a x = 0;$$

(v) *if there exists $g : \mathbb{T} \to \mathbb{R}$ with $|f(x)| \le g(x)$ for all $x \in [a,b]$ and $x^\alpha > a$, then*

$$\left|\int\limits_\tau^b f(x)\,\Delta_\alpha^a x\right| \le \int\limits_\tau^b g(x)\,\Delta_\alpha^a x;$$

(vi) *if $f(x) > 0$ for all $x \in [\tau, b]$ and $x^\alpha > a$, then*

$$\int\limits_\tau^b f(x)\,\Delta_\alpha^a x \ge 0.$$

Theorem 4.3. *If $f : \mathbb{T}^\kappa \to \mathbb{R}$ is an rd-continuous function and $x \in \mathbb{T}^\kappa$, then*

$$\int_x^{\sigma(t)} f(y)\,\Delta_\alpha^a y = \mu(x)\,\frac{x^\alpha - a}{x - a}\,f(x).$$

Proof. The proof is similar to that of Theorem 32 in [4]. □

Theorem 4.4. *Let $\mathbb{T}$ be a time scale, $a, b \in \mathbb{T}$ with $a < b$. Let $f : \mathbb{T} \to \mathbb{R}$ be a function.*

If $\Delta_a^{(\alpha)}(f)(x) \geq 0$ for all $x \in [a,b]_\mathbb{T}$ and $x^\alpha > a$, then f is an increasing function on $[a,b]_\mathbb{T}$.

If $\Delta_a^{(\alpha)}(f)(x) \leq 0$ for all $x \in [a,b]_\mathbb{T}$ and $x^\alpha > a$, then f is a decreasing function on $[a,b]_\mathbb{T}$.

Proof. The proof is similar to that of Theorem 33 in [4]. □

5 An application

In this section, we are concerned with the existence of a solution for the following $\Delta_a^{(\alpha)}$-conformable fractional nonlocal thermistor problem on time scales:

$$\begin{cases} \Delta_a^{(\alpha)}(x)(t) = \dfrac{\lambda f(t, x^\sigma(t))}{\left(\int_a^b f(\tau, x^\sigma(\tau))\,\Delta\tau\right)^2} \quad \text{for all } t \in \mathbb{T}_0^\kappa, \\ x(a) = x_0, \end{cases} \tag{5.1}$$

where $\mathbb{T}$ is an arbitrary bounded time scale such that $0 < a = \min \mathbb{T} < b = \max \mathbb{T}, \mathbb{T}_0 = \mathbb{T} \setminus \{a\}$, λ, x_0 are the fixed positive reals and $f : \mathbb{T}^\kappa \times \mathbb{R}^+ \to \mathbb{R}_+^*$ is a continuous function, x describes the temperature of the conductor and $\Delta_a^{(\alpha)}(x)(t)$ denotes the $\Delta_a^{(\alpha)}$-conformable fractional derivative of f of order $\alpha \in (0,1)$ at t. A solution of problem (5.1) will be a function $x \in C_{rd}^\alpha(\mathbb{T}, \mathbb{R})$ for which (5.1) is satisfied.

To prove our existence theorems, we need the following auxiliary lemma.

Lemma 5.1. *Let $\alpha \in (0,1)$. Problem (5.1) is equivalent to the following integral equation:*

$$x(t) = x_0 + \int_a^t \frac{s^\alpha - a}{s - a}\,\frac{\lambda f(s, x^\sigma(s))}{\left(\int_a^b f(\tau, x^\sigma(\tau))\,\Delta\tau\right)^2}\,\Delta s.$$

Proof. Suppose that x verifies (5.1). By Definitions 4.1 and 4.3, we have

$$\int_a^t (\Delta_a^{(\alpha)}(x)(s))\,\Delta_\alpha^a s = x(t) - x(a)$$

and

$$\int_a^t (\Delta_a^{(\alpha)}(x)(s))\,\Delta_\alpha^a s = \int_a^t \frac{\lambda f(s, x^\sigma(s))}{\left(\int_a^b f(\tau, x^\sigma(\tau))\,\Delta\tau\right)^2}\,\Delta_\alpha^a s,$$

then we get

$$x(t) - x(a) = \int_a^t \frac{\lambda f(s, x^\sigma(s))}{\left(\int_a^b f(\tau, x^\sigma(\tau))\,\Delta\tau\right)^2}\,\Delta_\alpha^a s.$$

So,

$$x(t) = x_0 + \int_a^t \frac{\lambda f(s, x^\sigma(s))}{\big(\int_a^b f(\tau, x^\sigma(\tau))\,\Delta\tau\big)^2}\,\Delta_\alpha^a s = x_0 + \int_a^t \frac{s^\alpha - a}{s-a}\,\frac{\lambda f(s, x^\sigma(s))}{\big(\int_a^b f(\tau, x^\sigma(\tau))\,\Delta\tau\big)^2}\,\Delta s. \qquad \square$$

Our first existence result for (5.1) is based on the Schauder fixed point theorem.

Theorem 5.1. *Let $f : \mathbb{T}^\kappa \times \mathbb{R}_+ \to \mathbb{R}_+^*$ be a continuous bounded function such that there exist two constants $m, M \in \mathbb{R}_+^*$ with $0 < m \le f(t,x) \le M$ for all $(t,x) \in (\mathbb{T}^\kappa, \mathbb{R}^+)$. Then problem (5.1) has a solution $x \in C_{rd}^\alpha(\mathbb{T}, \mathbb{R})$.*

Proof. We transform problem (5.1) into the fixed point problem. Let us define the operator $\mathcal{N} : C(\mathbb{T}, \mathbb{R}) \to C(\mathbb{T}, \mathbb{R})$ by

$$\mathcal{N}(x)(t) = x_0 + \int_a^t \frac{s^\alpha - a}{s-a}\,\frac{\lambda f(s, x^\sigma(s))}{\big(\int_a^b f(\tau, x^\sigma(\tau))\,\Delta\tau\big)^2}\,\Delta s. \tag{5.2}$$

From Lemma 5.1, the fixed point of the operator $\mathcal{N}$ is a solution of problem (5.1). We will show that $\mathcal{N}$ satisfies the assumptions of Theorem 2.3.

Step 1: $\mathcal{N}$ is continuous.

Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence of $C(\mathbb{T}, \mathbb{R})$ converging to $x \in C(\mathbb{T}, \mathbb{R})$. In this case, it is clear that

$$\begin{aligned}
&\big|\mathcal{N}(x_n(t)) - \mathcal{N}(x(t))\big| \\
&\le \lambda \int_a^t \Big|\frac{s^\alpha - a}{s-a}\Big| \left| \frac{f(s, x_n^\sigma(s))}{\big(\int_a^b f(\tau, x_n^\sigma(\tau))\,\Delta\tau\big)^2} - \frac{f(s, x^\sigma(s))}{\big(\int_a^b f(\tau, x^\sigma(\tau))\,\Delta\tau\big)^2} \right| \Delta s \\
&\le \lambda \int_a^t \left| \frac{f(s, x_n^\sigma(s)) - f(s, x^\sigma(s))}{\big(\int_a^b f(\tau, x_n^\sigma(\tau))\,\Delta\tau\big)^2} + \frac{f(s, x^\sigma(s))}{\big(\int_a^b f(\tau, x_n^\sigma(\tau))\,\Delta\tau\big)^2} - \frac{f(s, x^\sigma(s))}{\big(\int_a^b f(\tau, x^\sigma(\tau))\,\Delta\tau\big)^2} \right| \Delta s \\
&= \lambda \int_a^t \left| \frac{f(s, x_n^\sigma(s)) - f(s, x^\sigma(s))}{\big(\int_a^b f(\tau, x_n^\sigma(\tau))\,\Delta\tau\big)^2} - f(s, x^\sigma(s))\,\frac{\big(\int_a^b f(\tau, x_n^\sigma(\tau))\,\Delta\tau\big)^2 - \big(\int_a^b f(\tau, x^\sigma(\tau))\,\Delta\tau\big)^2}{\big(\int_a^b f(\tau, x_n^\sigma(\tau))\,\Delta\tau\big)^2\big(\int_a^b f(\tau, x^\sigma(\tau))\,\Delta\tau\big)^2} \right| \Delta s \\
&\le \lambda \int_a^t \frac{1}{\big(\int_a^b f(\tau, x_n^\sigma(\tau))\,\Delta\tau\big)^2\big(\int_a^b f(\tau, x^\sigma(\tau))\,\Delta\tau\big)^2} \\
&\qquad\qquad \times \left[\Big| f(s, x_n(\sigma(s))) - f(s, x(\sigma(s))) \Big| \left(\int_a^b f(\tau, x^\sigma(\tau))\,\Delta\tau \right)^2 \right. \\
&\qquad + \big| f(s, x(\sigma(s))) \big| \left(\int_a^b \Big| f(\tau, x_n(\sigma(\tau))) - f(\tau, x(\sigma(\tau))) \Big|\,\Delta\tau \right) \\
&\qquad\qquad \left. \times \left(\int_a^b \Big| f(\tau, x_n(\sigma(\tau))) + f(\tau, x(\sigma(\tau))) \Big|\,\Delta\tau \right) \right] \Delta s \\
&\le \frac{\lambda M^2}{m^4 (b-a)^3} \int_a^t \Big((b-a) \Big| f(s, x_n(\sigma(s))) - f(s, x(\sigma(s))) \Big|
\end{aligned}$$

$$+\,2\int_a^b \Big|f(\tau,x_n(\sigma(\tau)))-f(\tau,x(\sigma(\tau)))\Big|\,\Delta\tau\Big)\,\Delta s. \tag{5.3}$$

Since there is a constant $R>0$ such that $\|x\|_{C(\mathbb{T},\mathbb{R})}<R$, there exists an index N such that $\|x_n\|_{C(\mathbb{T},\mathbb{R})}\leq R$ for all $n>N$. Thus f is uniformly continuous on $\mathbb{T}^\kappa\times S_R(0)$ with $S_R(0)=\{x\in C(\mathbb{T},\mathbb{R}):\ \|x\|\leq R\}$. Therefore, for the given $\epsilon>0$, there is $\delta>0$ such that for all $x,y\in\mathbb{R}$, where

$$|y-x|<\delta<\frac{m^4(b-a)\varepsilon}{3\lambda M^2}\,,$$

one has

$$\big|f(s,y)-f(s,x)\big|<\frac{m^4(b-a)\varepsilon}{3\lambda M^2}\ \text{ for all }\ s\in\mathbb{T}^\kappa.$$

By assumption, one can find an index $\widehat{N}>N$ such that $\|x_n-x\|_{C(\mathbb{T},\mathbb{R})}<\delta$ for $n>\widehat{N}$. In this case,

$$\begin{aligned}\big\|\mathcal{N}(x_n(t))-\mathcal{N}(x(t))\big\| &\leq \frac{\lambda M^2}{m^4(b-a)^3}\int_a^t\Big((b-a)\,\frac{m^4(b-a)\varepsilon}{3\lambda M^2}+2\,\frac{m^4(b-a)\varepsilon}{3\lambda M^2}\,(b-a)\Big)\,\Delta s\\ &\leq \varepsilon.\end{aligned}$$

This proves the continuity of $\mathcal{N}$.

Step 2: The set $\mathcal{N}(C(\mathbb{T},\mathbb{R}))$ is relatively compact.

Consider a sequence $\{y_n\}_{n\in\mathbb{N}}$ of $\mathcal{N}(C(\mathbb{T},\mathbb{R}))$ for all $n\in\mathbb{N}$. There is $x_n\in C(\mathbb{T},\mathbb{R})$ such that $y_n=\mathcal{N}(x_n)$. We have

$$\begin{aligned}|\mathcal{N}(x_n)(t)| &\leq |x_0|+\lambda\int_a^t\Big|\frac{s^\alpha-a}{s-a}\Big|\,\frac{|f(s,x_n^\sigma(s))|}{\big(\int_a^b f(\tau,x_n^\sigma(\tau))\,\Delta\tau\big)^2}\,\Delta s\\ &\leq |x_0|+\frac{\lambda}{m^2(b-a)^2}\int_a^t\big|f(s,x_n(\sigma(s)))\big|\,\Delta s\\ &\leq |x_0|+\frac{\lambda M}{m^2(b-a)}\,.\end{aligned}$$

Thus $\|\mathcal{N}(x_n)(t)\|\leq|x_0|+\frac{\lambda M}{m^2(b-a)}=K$. So, $\mathcal{N}(C(\mathbb{T},\mathbb{R}))$ is uniformly bounded.
This set is also equicontinuous, since for every $t_1<t_2\in\mathbb{T}$,

$$\begin{aligned}&\big|\mathcal{N}(x_n)(t_2)-\mathcal{N}(x_n)(t_1)\big|\\ &\quad=\Big|\int_a^{t_2}\frac{s^\alpha-a}{s-a}\,\frac{\lambda f(s,x^\sigma(s))}{\big(\int_a^b f(\tau,x^\sigma(\tau))\,\Delta\tau\big)^2}\,\Delta s-\int_a^{t_1}\frac{s^\alpha-a}{s-a}\,\frac{\lambda f(s,x^\sigma(s))}{\big(\int_a^b f(\tau,x^\sigma(\tau))\,\Delta\tau\big)^2}\,\Delta s\Big|\\ &\quad=\Big|\int_{t_1}^{t_2}\frac{s^\alpha-a}{s-a}\,\frac{\lambda f(s,x^\sigma(s))}{\big(\int_a^b f(\tau,x^\sigma(\tau))\,\Delta\tau\big)^2}\,\Delta s\Big|\\ &\quad\leq\frac{\lambda}{m^2(b-a)^2}\int_{t_1}^{t_2}\big|f(s,x_n(\sigma(s)))\big|\,\Delta s\\ &\quad\leq\frac{\lambda M}{m^2(b-a)^2}\,|t_2-t_1|.\end{aligned}$$

The right-hand side tends to zero as $t_2 \to t_1$.

By the Arzelà–Ascoli theorem, we conclude that the set $\mathcal{N}(C(\mathbb{T},\mathbb{R}))$ is relatively compact in $C(\mathbb{T},\mathbb{R})$. Hence $\mathcal{N}$ is compact. As a consequence, the Schauder fixed-point theorem ensures that the operator $\mathcal{N}$ has a fixed point, which is a solution of problem (5.1). □

The next existence result is based on the Banach fixed point theorem.

Theorem 5.2. *Let $f : \mathbb{T}^\kappa \times \mathbb{R}^+ \to \mathbb{R}_+^*$ be a continuous bounded function such that there exist $m, M \in \mathbb{R}_+^*$ with $0 < m \le f(t,x) \le M$ for all $(t,x) \in (\mathbb{T}^\kappa, \mathbb{R}^+)$ and*

$$\exists\, L > 0,\ \ |f(t,u) - f(t,v)| \le L\|u - v\| \ \text{ for any } \ u, v \in \mathbb{R} \ \text{ and } \ t \in \mathbb{T}^\kappa.$$

If $\frac{m^4(b-a)}{3\lambda L M^2} > 1$, then problem (5.1) has a unique solution $x \in C_{rd}^{\alpha}(\mathbb{T},\mathbb{R})$.

Proof. We use the Banach fixed point theorem to prove that $\mathcal{N}$ defined by (5.2) has a fixed point. Let $x, y \in C(\mathbb{T},\mathbb{R})$ and $t \in \mathbb{T}^\kappa$. Then from (5.3) we have

$$\begin{aligned}
&\big|\mathcal{N}(y(t)) - \mathcal{N}(x(t))\big| \\
&\le \lambda \int_a^t \Big|\frac{s^\alpha - a}{s - a}\Big| \left| \frac{f(s, y^\sigma(s))}{\big(\int_a^b f(\tau, y^\sigma(\tau))\,\Delta\tau\big)^2} - \frac{f(s, x^\sigma(s))}{\big(\int_a^b f(\tau, x^\sigma(\tau))\,\Delta\tau\big)^2} \right| \Delta s \\
&\le \lambda \int_a^t \left| \frac{f(s, y^\sigma(s)) - f(s, x^\sigma(s))}{\big(\int_a^b f(\tau, y^\sigma(\tau))\,\Delta\tau\big)^2} - f(s, x^\sigma(s))\, \frac{\big(\int_a^b f(\tau, y^\sigma(\tau))\,\Delta\tau\big)^2 - \big(\int_a^b f(\tau, x^\sigma(\tau))\,\Delta\tau\big)^2}{\big(\int_a^b f(\tau, y^\sigma(\tau))\,\Delta\tau\big)^2 \big(\int_a^b f(\tau, x^\sigma(\tau))\,\Delta\tau\big)^2} \right| \Delta s \\
&\le \frac{\lambda M^2}{m^4 (b-a)^3} \int_a^t \Big((b-a)\, \big|f(s, y(\sigma(s))) - f(s, x(\sigma(s)))\big| + 2\int_a^b \big|f(\tau, y(\sigma(\tau))) - f(\tau, x(\sigma(\tau)))\big|\,\Delta\tau \Big)\,\Delta s \\
&\le \frac{\lambda M^2}{m^4 (b-a)^3} \big(L(b-a)\|y - x\| + 2L(b-a)\|y - x\| \big)(b-a).
\end{aligned}$$

Thus

$$\big\|\mathcal{N}(y(t)) - \mathcal{N}(x(t))\big\| \le \frac{3\lambda L M^2}{m^4 (b-a)}\, \|y - x\|.$$

Since $0 < \frac{3\lambda L M^2}{m^4(b-a)} < 1$, $\mathcal{N}$ is a contraction. This implies that problem (5.1) has a unique solution $x \in C_{rd}^{\alpha}(\mathbb{T},\mathbb{R})$, by the Banach fixed point theorem. □

Example 5.1. Consider the following nonlinear conformable fractional thermistor problem:

$$\begin{cases} \Delta_1^{(\frac{1}{3})}(x)(t) = \dfrac{\frac{2t}{e^t+5} + \frac{1+|x(\sigma(t))|}{2+|x(\sigma(t))|}}{25\big(\int_1^2 \big(\frac{2\tau}{e^\tau+5} + \frac{1+|x(\sigma(\tau))|}{2+|x(\sigma(\tau))|}\big)\,\Delta\tau\big)^2}, \quad t \in I = (1,2)_{\mathbb{T}}, \\ x(1) = 0.5. \end{cases} \tag{5.4}$$

Here, $\alpha = \frac{1}{3}$, $a = 1$, $b = 2$, $\lambda = \frac{1}{25}$, $f(t,x) = \frac{2t}{e^t+5} + \frac{1+x}{2+x}$, $t \in [1,2)_{\mathbb{T}}$ and $x \in \mathbb{R}_+$. It is clear that f is a continuous function. For $t \in [1,2]_{\mathbb{T}}$ and $x, y \in \mathbb{R}_+$, we have

$$|f(t,x) - f(t,y)| = \Big|\frac{1+x}{2+x} - \frac{1+y}{2+y}\Big| = \Big|\frac{x - y}{4 + 2x + 2y + xy}\Big| \le \frac{1}{4}\,|x - y|.$$

We can easily see that

$$\frac{1}{7} \le \frac{2t}{e^t + 5} \le \frac{4}{5}, \quad \frac{1}{2} \le \frac{1+x}{2+x} = 1 - \frac{1}{2+x} \le 1$$

and

$$\frac{9}{14} \leq f(t,x) \leq \frac{9}{5}\,.$$

Then, all conditions of Theorem 5.2 are satisfied with $L = \frac{1}{4}$, $m = \frac{9}{14}$, $M = \frac{9}{5}$, $\frac{m^4(b-a)}{3\lambda L M^2} \simeq 1.76 > 1$, and we conclude that problem (5.4) has a unique solution $x \in C_{rd}^{\frac{1}{3}}([1,2]_{\mathbb{T}}, \mathbb{R}_+)$.

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