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EXISTENCE OF LIMIT CYCLES FOR A CERTAIN CLASS OF GENERALIZED PERTURBED KUKLES SYSTEMS VIA AVERAGING THEORY

Abstract. This research aims to give a recent result for the existence of limit cycles that can be bifurcated from the origin of coordinates in a certain class of generalized perturbed Kukles systems using averaging theory.

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1 Introduction

This work is related to the Kukles polynomial differential system [6]

$$\dot{x} = -y, \quad \dot{y} = P(x, y),$$

where the polynomial P has degree n. Note that the system coincides with the classical Kukles system introduced by Kukles [7] who studied a linear center with cubic non-homogeneous nonlinearities. J. Giné in [6] provided the conditions that are necessary and sufficient to prove the existence of linear centers in two different homogeneous systems nonlinearity, the first is a homogeneous quartic nonlinearity, and the other is a homogeneous quintic of nonlinearity. In [13], A. P. Sadovskii solved the problem of center-focus for the given system

$$\dot{x} = -y, \quad \dot{y} = x + a_0 y + a_1 x^2 + a_2 xy + a_3 y^2 + a_4 x^3 + a_5 x^2 y + a_6 xy^2 + a_7 y^3,$$

where a_i are real coefficients for each $i = 0, \ldots, 7$ with $a_2 a_7 \neq 0$, and demonstrated that the system can have up to seven limit cycles. Many authors have considered this conjecture, see, e.g. [5, 9, 12, 16]. Recently, the problem that persists in controlling the existence of limit cycles of some type of differential systems has a great importance. We call a limit cycle every isolated periodic solution in periodic solutions of differential system. To prove the existence of limit cycles various methods were used to obtain a maximum number of limit cycles that can be bifurcated from the periodic orbits of a linear center. These methods include the Poincaré return map, the inverse integrating factor, and the averaging theory. In [2], A. Belfar and R. Benterki used the averaging theory up to sixth order to search the existence of limit cycles that can be bifurcated from the origin of coordinates for the following perturbed Kukles system

$$\begin{cases} \dot{x} = -y + \sum_{s=1}^{6} \varepsilon^s \sum_{0 \leq i+j \leq 8} \alpha_{ij}^{(s)} x^i y^j, \\ \dot{y} = x + ax^8 + bx^4 y^4 + cy^8 + \sum_{s=1}^{6} \varepsilon^s \sum_{0 \leq i+j \leq 8} \beta_{ij}^{(s)} x^i y^j, \end{cases}$$

where $i, j \in \mathbb{N}$.

In [11], by applying the averaging theory, the authors found a result for the maximum number of limit cycles for the given class of generalized perturbed Kukles system

$$\begin{cases} \dot{x} = -y + l(x), \\ \dot{y} = x - f(x) - g(x)y - h(x)y^2 - d_0 y^3, \end{cases}$$

where

$$l(x) = \varepsilon l^1(x) + \varepsilon^2 l^2(x), \quad f(x) = \varepsilon f^1(x) + \varepsilon^2 f^2(x), \quad g(x) = \varepsilon g^1(x) + \varepsilon^2 g^2(x),$$
$$h(x) = \varepsilon h^1(x) + \varepsilon^2 h^2(x), \quad d_0 = \varepsilon d_0^1(x) + \varepsilon^2 d_0^2(x),$$

and $l^k(x)$, $f^k(x)$, $g^k(x)$ and $h^k(x)$ have degree m, n_1, n_2 and n_3, respectively, $d_0^k = 0$ is a real number for every $k = 1, 2$, and ε is a small parameter.

In [3], A. Boulfoul, N. Mellahi and A. Makhlouf found new bounded limit cycles that can be bifurcated from the periodic orbits for the following class of perturbed kukles system:

$$\begin{cases} \dot{x} = -y, \\ \dot{y} = x - f(x) - g(x)y - h(x)y^2 - l(x)y^3, \end{cases}$$

where

$$f(x) = \varepsilon f_1(x) + \varepsilon^2 f_2(x), \quad g(x) = \varepsilon g_1(x) + \varepsilon^2 g_2(x), \quad h(x) = \varepsilon h_1(x) + \varepsilon^2 h_2(x), \quad l(x) = \varepsilon l_1(x) + \varepsilon^2 l_2(x)$$

have degree n_1, n_2, n_3 and n_4, respectively, and ε is a small parameter.

In [10], the authors applied the averaging theory to research the number of limit cycles for the following class of generalized perturbed Kukles system:

$$\begin{cases} \dot{x} = y, \\ \dot{y} = -x - f_1(x,y) - f_2(x,y)y - g_1(x,y)y^2 - g_2(x,y)y^3, \end{cases}$$

where

$$f_1(x,y) = \varepsilon f_{11}(x,y) + \varepsilon^2 f_{12}(x,y), \quad f_2(x,y) = \varepsilon f_{21}(x,y) + \varepsilon^2 f_{22}(x,y),$$
$$g_1(x,y) = \varepsilon g_{11}(x,y) + \varepsilon^2 g_{12}(x,y), \quad g_2(x,y) = \varepsilon g_{21}(x,y) + \varepsilon^2 g_{22}(x,y)$$

have degree n_1, n_2, n_3 and n_4, respectively, and ε is a small parameter.

The principal objective of our research is to give a recent result about the maximum number of limit cycles that can be bifurcated from the periodic orbits of $\dot{x} = -y$, $\dot{y} = x$ via the averaging theory for the following class of generalized perturbed Kukles system:

$$\begin{cases} \dot{x} = -y + L(x,y) + P(x,y)y, \\ \dot{y} = x - F(x,y) - G(x,y)y - H(x,y)y^2 - Q(x,y)y^3, \end{cases} \tag{1.1}$$

where

$$L(x,y) = \varepsilon L_1(x,y) + \varepsilon^2 L_2(x,y), \quad P(x,y) = \varepsilon P_1(x,y) + \varepsilon^2 P_2(x,y),$$
$$F(x,y) = \varepsilon F_1(x,y) + \varepsilon^2 F_2(x,y), \quad G(x,y) = \varepsilon G_1(x,y) + \varepsilon^2 G_2(x,y),$$
$$H(x,y) = \varepsilon H_1(x,y) + \varepsilon^2 H_2(x,y), \quad Q(x,y) = \varepsilon Q_1(x,y) + \varepsilon^2 Q_2(x,y),$$

and L_i, P_i, F_i, G_i, H_i and Q_i have degree m_1, m_2, n_1, n_2, n_3 and n_4, respectively, for every $i = 1, 2$, and ε is a small parameter.

The following theorems show our main result.

Theorem 1.1. *The maximum number of limit cycles bifurcating from the periodic orbits of linear center $\dot{x} = -y$, $\dot{y} = x$, obtained by applying the averaging method of the first order and by perturbations within the family of generalized perturbed Kukles system (1.1), is*

$$\max\left\{\left[\frac{m_1-1}{2}\right], \left[\frac{m_2}{2}\right], \left[\frac{n_1-1}{2}\right], \left[\frac{n_2}{2}\right], \left[\frac{n_3+1}{2}\right], \left[\frac{n_4+2}{2}\right]\right\}.$$

Theorem 1.2. *The maximum number of limit cycles bifurcating from the periodic orbits of linear center $\dot{x} = -y$, $\dot{y} = x$, obtained by applying the averaging method of the second order and by perturbations within the family of generalized perturbed Kukles system (1.1), is*

$$\max\left\{\left[\frac{E(m_1)+E(n_1)}{2}\right], O(n_4)+3, \eta_1+1, \eta_2+2, \eta_3+3\right\},$$

where

$$\begin{aligned}
\eta_1 = \max\Big\{ & \left[\frac{O(m_1)-1}{2}\right], \left[\frac{O(n_1)-1}{2}\right], \left[\frac{E(n_2)}{2}\right], E(m_1)-1, E(n_1)-1, \\
& \left[\frac{E(m_1)+O(n_2)-1}{2}\right], \left[\frac{E(m_1)+E(n_3)}{2}\right], \left[\frac{E(n_1)+O(n_2)-1}{2}\right], \\
& \left[\frac{O(m_2)+E(n_1)-1}{2}\right], \left[\frac{O(m_2)+E(m_1)-1}{2}\right]\Big\},
\end{aligned}$$

$$\begin{aligned}
\eta_2 = \max\Big\{ & \left[\frac{O(n_3)-1}{2}\right], \left[\frac{E(n_4)}{2}\right], \left[\frac{E(m_2)-2}{2}\right], \left[\frac{E(n_1)+E(n_3)-2}{2}\right], \\
& \left[\frac{E(n_1)+O(n_4)-1}{2}\right], O(n_2)-1, \left[\frac{O(n_2)+E(n_3)-1}{2}\right], \left[\frac{E(m_1)+O(n_4)-1}{2}\right], \\
& \left[\frac{O(m_2)+O(n_2)-2}{2}\right], \left[\frac{O(m_2)+E(n_3)-1}{2}\right], O(m_2)-1\Big\},
\end{aligned}$$

$$\eta_3 = \max\left\{\left[\frac{O(n_2)+O(n_4)-2}{2}\right], E(n_3)-1, \left[\frac{E(n_3)+O(n_4)-1}{2}\right], \left[\frac{O(m_2)+O(n_4)-2}{2}\right]\right\},$$

here $E(i)$ represents the biggest even integer that $\leq i$, $O(i)$ represents the biggest odd integer that $\leq i$.

The proof of Theorem 1.1 is given in Section 3. The proof of Theorem 1.2 is given in Section 4. The averaging theory is the main tool used to prove the results of Theorem 1.1 and Theorem 1.2. This will be discussed in section 2.

2 The averaging theory of first and second order

In this section, we summarize the first and second order averaging theory developed in [4] and [8] in the following theorem.

Theorem 2.1. *Let the differential system*

$$x'(t) = \varepsilon f_1(t,x) + \varepsilon^2 f_2(t,x) + \varepsilon^3 r(t,x,\varepsilon) \tag{2.1}$$

be given, where $f_1, f_2 : \mathbb{R}\times D \to \mathbb{R}^n$, $r : \mathbb{R}\times D\times(-\varepsilon_f, \varepsilon_f) \to \mathbb{R}^n$ are continuous functions, T-periodic in the first variable, and D is an open subset of $\mathbb{R}^n$. Suppose that the following hypotheses hold:

(i) *$f_1(t,\cdot) \in C^1(D)$ for all $t\in\mathbb{R}$, f_2, R, D_xF_1 are locally Lipschitz with respect to x, and R is differentiable with respect to ε. Define $f_{h0} : D \to \mathbb{R}^n$ for $h = 1, 2$ as*

$$f_{10}(z) = \frac{1}{T}\int_0^T f_1(s,z)\,ds,$$

$$f_{20}(z) = \frac{1}{T}\int_0^T \left[D_z f_1(s,z)\cdot y_1(s,z) + f_2(s,z)\right] ds,$$

with

$$y_1(s,z) = \int_0^s f_1(t,z)\,dt,$$

(ii) *For an open and bounded set $V \subset D$ and for every $\varepsilon \in (-\varepsilon_f, \varepsilon_f)\setminus\{0\}$, there exists $a_\epsilon \in V$ such that*

$$f_{10}(a_\varepsilon) + \varepsilon f_{20}(a_\varepsilon) = 0 \;\; \text{and} \;\; d_B(f_{10} + \varepsilon f_{20}, V, a_\varepsilon) \neq 0.$$

Then for sufficiently small $|\varepsilon| > 0$ there exists a T-periodic solution $\varphi(\cdot,\varepsilon)$ of system (2.1) such that $\varphi(0,\varepsilon) = a_\varepsilon$.

The expression $d_B(f_{10}+\varepsilon f_{20}, V, a_\varepsilon) \neq 0$ means that the Brouwer degree of the function $f_{10}+\varepsilon f_{20} : V \to \mathbb{R}^n$ at the fixed point a_ε is not zero. A sufficient condition for the inequality to be true is that the Jacobian of the function $f_{10} + \varepsilon f_{20}$ at a_ε is not zero.

If f_{10} is not identically zero, then the roots of $f_{10}+\varepsilon f_{20}$ are mainly the roots of f_{10} for ε sufficiently small. In this case, the previous result provides the *first order averaging theory.*

If f_{10} is identically zero and f_{20} is not identically zero, then the roots of $f_{10} + \varepsilon f_{20}$ are mainly the roots of f_{20} for ε sufficiently small. In this case, the previous result provides the *second order averaging theory.*

More information on the averaging theory can be found in [14] and [15].

3 Proof of Theorem 1.1

To prove the result of Theorem 1.1, we apply the first order averaging theory presented in Section 2. After we consider system (1.1) and take the following functions

$$F_1(x,y) = \sum_{i+j=0}^{n_1} a_{ij,1}x^iy^j, \quad L_1(x,y) = \sum_{i+j=0}^{m_1} b_{ij,1}x^iy^j, \quad P_1(x,y) = \sum_{i+j=0}^{m_2} p_{ij,1}x^iy^j,$$

$$G_1(x,y)=\sum_{i+j=0}^{n_2} c_{ij,1}x^iy^j,\quad H_1(x,y)=\sum_{i+j=0}^{n_3} d_{ij,1}x^iy^j,\quad Q_1(x,y)=\sum_{i+j=0}^{n_4} e_{ij,1}x^iy^j.$$

Changing to polar coordinates $x=r\cos(\phi)$, $y=r\sin(\phi)$ and $r>0$, we find the system for $(\dot{r},\dot{\phi})$. Taking ϕ as a new independent variable, we get the differential equation for $\dot{r}/\dot{\phi}$. Therefore, system (1.1) becomes

$$\frac{dr}{d\phi}=\varepsilon f_1(r,\phi)+O(\varepsilon^2), \tag{3.1}$$

where

$$\begin{aligned} f_1(r,\phi)=&\sum_{i+j=0}^{m_1} b_{ij,1}r^{i+j}\cos^{i+1}(\phi)\sin^j(\phi)+\sum_{i+j=0}^{m_2} p_{ij,1}r^{i+j+1}\cos^{i+1}(\phi)\sin^{j+1}(\phi)\\ &-\sum_{i+j=0}^{n_1} a_{ij,1}r^{i+j}\cos^i(\phi)\sin^{j+1}(\phi)-\sum_{i+j=0}^{n_2} c_{ij,1}r^{i+j+1}\cos^i(\phi)\sin^{j+2}(\phi)\\ &-\sum_{i+j=0}^{n_3} d_{ij,1}r^{i+j+2}\cos^i(\phi)\sin^{j+3}(\phi)-\sum_{i+j=0}^{n_4} e_{ij,1}r^{i+j+3}\cos^i(\phi)\sin^{j+4}(\phi).\end{aligned}$$

Applying the first order averaging theory, we calculate

$$f_{10}=\frac{1}{2\pi}\int_0^{2\pi} f_1(r,\phi)\,d\phi.$$

Then, using the formulas

$$\begin{aligned}
&\int_0^{2\pi}\cos^{i+1}(\phi)\sin^j(\phi)\,d\phi=\begin{cases}\pi\alpha_{ij} & \text{if } i=2e+1,\ \ j=2e\ \ (e\in\mathbb{N}),\\ 0 & \text{if not},\end{cases}\\
&\int_0^{2\pi}\cos^{i}(\phi)\sin^{j+1}(\phi)\,d\phi=\begin{cases}\pi\beta_{ij} & \text{if } i=2e,\ \ j=2e+1\ \ (e\in\mathbb{N}),\\ 0 & \text{if not},\end{cases}\\
&\int_0^{2\pi}\cos^{i}(\phi)\sin^{j+2}(\phi)\,d\phi=\begin{cases}\pi\gamma_{ij} & \text{if } i=2e,\ \ j=2e\ \ (e\in\mathbb{N}),\\ 0 & \text{if not},\end{cases}\\
&\int_0^{2\pi}\cos^{i}(\phi)\sin^{j+3}(\phi)\,d\phi=\begin{cases}\pi\varsigma_{ij} & \text{if } i=2e,\ \ j=2e+1\ \ (e\in\mathbb{N}),\\ 0 & \text{if not},\end{cases}\\
&\int_0^{2\pi}\cos^{i}(\phi)\sin^{j+4}(\phi)\,d\phi=\begin{cases}\pi\delta_{ij} & \text{if } i=2e,\ \ j=2e\ \ (e\in\mathbb{N}),\\ 0 & \text{if not},\end{cases}\\
&\int_0^{2\pi}\cos^{i+1}(\phi)\sin^{j+1}(\phi)\,d\phi=\begin{cases}\pi\xi_{ij} & \text{if } i=2e+1,\ \ j=2e+1\ \ (e\in\mathbb{N}),\\ 0 & \text{if not},\end{cases}
\end{aligned} \tag{3.2}$$

where α_{ij}, β_{ij}, γ_{ij}, ς_{ij}, δ_{ij}, and ξ_{ij} are constants different from zero, we find

$$\begin{aligned} f_{10}(r)=\frac{r}{2}\Bigg[&\sum_{i+j=0}^{[\frac{m_1-1}{2}]} b_{(2i+1)(2j),1}\alpha_{ij}r^{2(i+j)}+\sum_{i+j=0}^{[\frac{m_2-2}{2}]} p_{(2i+1)(2j+1),1}\xi_{ij}r^{2(i+j)+2}\\ &-\sum_{i+j=0}^{[\frac{n_1-1}{2}]} a_{(2i)(2j+1),1}\beta_{ij}r^{2(i+j)}-\sum_{i+j=0}^{[\frac{n_2}{2}]} c_{(2i)(2j),1}\gamma_{ij}r^{2(i+j)}\end{aligned}$$

$$- \sum_{i+j=0}^{[\frac{n_3-1}{2}]} d_{(2i)(2j+1),1}\varsigma_{ij} r^{2(i+j)+2} - \sum_{i+j=0}^{[\frac{n_4}{2}]} e_{(2i)(2j),1}\delta_{ij} r^{2(i+j)+2}\Bigg].$$

To have the maximum number of positive real zeros, we calculate $f_{10}(r) = 0 \Longleftrightarrow r = 0$, or

$$\sum_{i+j=0}^{[\frac{m_1-1}{2}]} b_{(2i)(2j+1),1}\alpha_{ij} r^{2(i+j)} + \sum_{i+j=0}^{[\frac{m_2-2}{2}]} p_{(2i+1)(2j+1),1}\xi_{ij} r^{2(i+j)+2} - \sum_{i+j=0}^{[\frac{n_1-1}{2}]} a_{(2i)(2j+1),1}\beta_{ij} r^{2(i+j)}$$
$$- \sum_{i+j=0}^{[\frac{n_2}{2}]} c_{(2i)(2j),1}\gamma_{ij} r^{2(i+j)} - \sum_{i+j=0}^{[\frac{n_3-1}{2}]} d_{(2i)(2j+1),1}\varsigma_{ij} r^{2(i+j)+2} - \sum_{i+j=0}^{[\frac{n_4}{2}]} e_{(2i)(2j),1}\delta_{ij} r^{2(i+j)+2} = 0.$$

Since we know that r must be a positive real number different from 0, we should look for the number of positive real zeros. So, we put

$$\sum_{i+j=0}^{[\frac{m_1-1}{2}]} b_{(2i)(2j+1),1}\alpha_{ij} r^{2(i+j)} + \sum_{i+j=0}^{[\frac{m_2-2}{2}]} p_{(2i+1)(2j+1),1}\xi_{ij} r^{2(i+j)+2} - \sum_{i+j=0}^{[\frac{n_1-1}{2}]} a_{(2i)(2j+1),1}\beta_{ij} r^{2(i+j)}$$
$$- \sum_{i+j=0}^{[\frac{n_2}{2}]} c_{(2i)(2j),1}\gamma_{ij} r^{2(i+j)} - \sum_{i+j=0}^{[\frac{n_3-1}{2}]} d_{(2i)(2j+1),1}\varsigma_{ij} r^{2(i+j)+2} - \sum_{i+j=0}^{[\frac{n_4}{2}]} e_{(2i)(2j),1}\delta_{ij} r^{2(i+j)+2} = 0$$
$$\Longrightarrow b_{01,1}\alpha_{00} + (b_{21,1}\alpha_{10} + b_{03,1}\alpha_{01})r^2 + (b_{05,1}\alpha_{02} + b_{41,1}\alpha_{20} + b_{23,1}\alpha_{11})r^4 + \cdots + b_{(2i)(2j+1),1}\alpha_{ij} r^{m_1-1}$$
$$+ p_{11,1}\xi_{00} r^2 + (p_{13,1}\xi_{01} + p_{31,1}\xi_{10})r^4 + \cdots + p_{(2i+1)(2j+1),1}\xi_{ij} r^{m_2}$$
$$- \Big(a_{01,1}\beta_{00} + (a_{03,1}\beta_{01} + a_{21,1}\beta_{10})r^2 + (a_{05,1}\beta_{02} + a_{41,1}\beta_{20} + a_{23,1}\beta_{11})r^4 + \cdots + a_{(2i)(2j+1),1}\beta_{ij} r^{n_1-1}\Big)$$
$$- \Big(c_{00,1}\gamma_{00} + (c_{02,1}\gamma_{01} + c_{20,1}\gamma_{10})r^2 + (c_{04,1}\gamma_{02} + c_{40,1}\gamma_{20} + c_{22,1}\gamma_{11})r^4 + \cdots + c_{(2i)(2j),1}\gamma_{ij} r^{n_2}\Big)$$
$$- \Big(d_{01,1}\varsigma_{00} r^2 + (d_{03,1}\varsigma_{01} + d_{21,1}\varsigma_{10})r^4 + \cdots + d_{(2i)(2j+1),1}\varsigma_{ij} r^{n_3+1}\Big)$$
$$- \Big(e_{00,1}\delta_{00} r^2 + (e_{02,1}\delta_{01} + e_{20,1}\delta_{10})r^4 + e_{(2i)(2j),1}\delta_{ij} r^{n_4+2}\Big) = 0.$$

The coefficients of each polynomial are related to the coefficients of the polynomial system (1.1) and the value of the integrals (3.2), which are already known. We can choose the coefficients of the polynomials of system (1.1) in such a way that the equation $f_{10}(r) = 0$ may have at most

$$\max\Big\{\Big[\frac{m_1-1}{2}\Big], \Big[\frac{m_2}{2}\Big], \Big[\frac{n_1-1}{2}\Big], \Big[\frac{n_2}{2}\Big], \Big[\frac{n_3+1}{2}\Big], \Big[\frac{n_4+2}{2}\Big]\Big\}$$

positive real zeros. Thus, the proof of Theorem 1.1 is complete.

4 Proof of Theorem 1.2

In the proof of Theorem (1.2), we apply the second order averaging theory to system (1.1). So, if we take $L_1(x,y)$, $P_1(x,y)$, $F_1(x,y)$, $G_1(x,y)$, $H_1(x,y)$ and $Q_1(x,y)$ presented in Section 3 and

$$F_2(x,y) = \sum_{i+j=0}^{n_1} a_{ij,2}x^i y^j, \quad L_2(x,y) = \sum_{i+j=0}^{m_1} b_{ij,2}x^i y^j, \quad P_2(x,y) = \sum_{i+j=0}^{m_2} p_{ij,2}x^i y^j,$$
$$G_2(x,y) = \sum_{i+j=0}^{n_2} c_{ij,2}x^i y^j, \quad H_2(x,y) = \sum_{i+j=0}^{n_3} d_{ij,2}x^i y^j, \quad Q_2(x,y) = \sum_{i+j=0}^{n_4} e_{ij,2}x^i y^j,$$

then we rewrite system (1.1) in polar coordinates (r,ϕ), $r > 0$, and find the differential system for $(\dot{r}, \dot{\phi})$. After that, we take ϕ as a new independent variable and obtain

$$\frac{dr}{d\phi} = \varepsilon f_1(r,\phi) + \varepsilon^2 f_2(r,\phi) + O(\varepsilon^3), \tag{4.1}$$

where

$$\begin{cases} f_1(r,\phi) = A, \\ f_2(r,\phi) = B + \dfrac{1}{r^2}\, AC, \end{cases}$$

and

$$\begin{aligned} A = &\sum_{i+j=0}^{m_1} b_{ij,1} r^{i+j} \cos^{i+1}(\phi)\sin^j(\phi) + \sum_{i+j=0}^{m_2} p_{ij,1} r^{i+j+1} \cos^{i+1}(\phi)\sin^{j+1}(\phi) \\ &- \sum_{i+j=0}^{n_1} a_{ij,1} r^{i+j} \cos^{i}(\phi)\sin^{j+1}(\phi) - \sum_{i+j=0}^{n_2} c_{ij,1} r^{i+j+1} \cos^{i}(\phi)\sin^{j+2}(\phi) \\ &- \sum_{i+j=0}^{n_3} d_{ij,1} r^{i+j+2} \cos^{i}(\phi)\sin^{j+3}(\phi) - \sum_{i+j=0}^{n_4} e_{ij,1} r^{i+j+3} \cos^{i}(\phi)\sin^{j+4}(\phi), \\ B = &\sum_{i+j=0}^{m_1} b_{ij,2} r^{i+j} \cos^{i+1}(\phi)\sin^j(\phi) + \sum_{i+j=0}^{m_2} p_{ij,2} r^{i+j+1} \cos^{i+1}(\phi)\sin^{j+1}(\phi) \\ &- \sum_{i+j=0}^{n_1} a_{ij,2} r^{i+j} \cos^{i}(\phi)\sin^{j+1}(\phi) - \sum_{i+j=0}^{n_2} c_{ij,2} r^{i+j+1} \cos^{i}(\phi)\sin^{j+2}(\phi) \\ &- \sum_{i+j=0}^{n_3} d_{ij,2} r^{i+j+2} \cos^{i}(\phi)\sin^{j+3}(\phi) - \sum_{i+j=0}^{n_4} e_{ij,2} r^{i+j+3} \cos^{i}(\phi)\sin^{j+4}(\phi), \\ C = &\sum_{i+j=0}^{n_1} a_{ij,1} r^{i+j+1} \cos^{i+1}(\phi)\sin^j(\phi) + \sum_{i+j=0}^{n_2} c_{ij,1} r^{i+j+2} \cos^{i+1}(\phi)\sin^{j+1}(\phi) \\ &+ \sum_{i+j=0}^{n_3} d_{ij,1} r^{i+j+3} \cos^{i+1}(\phi)\sin^{j+2}(\phi) + \sum_{i+j=0}^{n_4} e_{ij,1} r^{i+j+4} \cos^{i+1}(\phi)\sin^{j+3}(\phi) \\ &+ \sum_{i+j=0}^{m_1} b_{ij,1} r^{i+j+1} \cos^{i}(\phi)\sin^{j+1}(\phi) + \sum_{i+j=0}^{m_2} p_{ij,1} r^{i+j+2} \cos^{i}(\phi)\sin^{j+2}(\phi). \end{aligned}$$

To use the second order averaging method, we must first cancel the first order averaged equation $f_{10}(r)$ introduced in Section 3. So, for $e \in \mathbb{N}$, we put

$$\begin{aligned} b_{ij,1} &= 0 \text{ for } i = 2e+1, \; j = 2e, \text{ where } i+j = 1,\dots,O(m_1), \\ p_{ij,1} &= 0 \text{ for } i = 2e+1, \; j = 2e+1, \text{ where } i+j = 2,\dots,E(m_2), \\ a_{ij,1} &= 0 \text{ for } i = 2e, \; j = 2e+1, \text{ where } i+j = 1,\dots,O(n_1), \\ c_{ij,1} &= 0 \text{ for } i = 2e, \; j = 2e, \text{ where } i+j = 0,\dots,E(n_2), \\ d_{ij,1} &= 0 \text{ for } i = 2e, \; j = 2e+1, \text{ where } i+j = 1,\dots,O(n_3), \\ e_{ij,1} &= 0 \text{ for } i = 2e, \; j = 2e, \text{ where } i+j = 0,\dots,E(n_4). \end{aligned}$$

We calculate the averaging function of the second order as follows:

$$f_{20}(r) = \frac{1}{2\pi}\int_0^{2\pi}\left[\frac{d}{dr}\, f_1(r,\phi)\int_0^{\phi} f_1(r,t)\,dt + f_2(r,\phi)\right] d\phi.$$

First, we determine the corresponding function

$$\frac{d}{dr}\, f_1(r,\phi) = \sum_{\substack{i+j=0 \\ i \text{ even or } j \text{ odd}}}^{m_1} (i+j) b_{ij,1} r^{i+j-1} \cos^{i+1}(\phi)\sin^j(\phi)$$

$$+ \sum_{\substack{i+j=0 \\ i \text{ even or } j \text{ even}}}^{m_2} (i+j+1)p_{ij,1} r^{i+j} \cos^{i+1}(\phi) \sin^{j+1}(\phi)$$
$$- \sum_{\substack{i+j=0 \\ i \text{ odd or } j \text{ even}}}^{n_1} (i+j)a_{ij,1} r^{i+j-1} \cos^{i}(\phi) \sin^{j+1}(\phi)$$
$$- \sum_{\substack{i+j=1 \\ i \text{ odd or } j \text{ odd}}}^{n_2} (i+j+1)c_{ij,1} r^{i+j} \cos^{i}(\phi) \sin^{j+2}(\phi)$$
$$- \sum_{\substack{i+j=0 \\ i \text{ odd or } j \text{ even}}}^{n_3} (i+j+2)d_{ij,1} r^{i+j+1} \cos^{i}(\phi) \sin^{j+3}(\phi)$$
$$- \sum_{\substack{i+j=1 \\ i \text{ odd or } j \text{ odd}}}^{n_4} (i+j+3)e_{ij,1} r^{i+j+2} \cos^{i}(\phi) \sin^{j+4}(\phi),$$

and

$$\int_0^\phi F_1(r,t)\,dt = b_{00,1}\sin(\phi) + b_{02,1}r^2\big(\alpha_{102}\sin(\phi) + \alpha_{202}\sin(3\phi)\big) + \cdots$$
$$+ b_{nm,1}r^{n+m}\Big(\alpha_{1nm}\sin(\phi) + \alpha_{2nm}\sin(3\phi) + \cdots + \alpha_{(\frac{n+m}{2}+2)nm}\sin((n+m+1)\phi)\Big)$$
$$+ b_{01,1}r(\alpha_{101} + \alpha_{201}\cos(2\phi)) + \cdots$$
$$+ b_{nv,1}r^{n+v}\Big(\alpha_{1nv} + \alpha_{2nv}\cos(2\phi) + \cdots + \alpha_{(\frac{n+v+1}{2}+3)nv}\cos((n+v+1)\phi)\Big)$$
$$+ b_{11,1}r^2\big(\alpha_{111} + \alpha_{211}\cos(\phi) + \alpha_{311}\cos(3\phi)\big) + \cdots$$
$$+ b_{uv,1}r^{u+v}\Big(\alpha_{1uv} + \alpha_{2uv}\cos(\phi) + \alpha_{3uv}\cos(3\phi) + \cdots + \alpha_{(\frac{u+v}{2}+2)uv}\Big)$$
$$+ a_{00,1}(\beta_{100} + \beta_{200}\cos(\phi)) + a_{02,1}r^2\big(\beta_{102} + \beta_{202}\cos(\phi) + \beta_{302}\cos(3\phi)\big) + \cdots$$
$$+ a_{nm,1}r^{n+m}\Big(\beta_{1nm} + \beta_{2nm}\cos(\phi) + \beta_{3nm}\cos(3\phi) + \cdots + \beta_{(\frac{n+m}{2}+2)nm}\cos((n+m+1)\phi)\Big)$$
$$+ a_{10,1}r(\beta_{110} + \beta_{210}\cos(2\phi)) + \cdots$$
$$+ a_{um,1}r^{u+m}\Big(\beta_{1um} + \beta_{2um}\cos(2\phi) + \cdots + \beta_{(\frac{u+m+1}{2}+1)um}\cos((u+m+1)\phi)\Big)$$
$$+ a_{11,1}r^2\big(\beta_{111}\sin(\phi) + \beta_{211}\sin(3\phi)\big) + \cdots$$
$$+ a_{uv,1}r^{u+v}\Big(\beta_{1uv}\sin(\phi) + \beta_{2uv}\sin(3\phi) + \cdots + \beta_{(\frac{u+v}{2}+1)uv}\sin((u+v+1)\phi)\Big)$$
$$+ c_{01,1}r^2\big(\gamma_{101} + \gamma_{201}\cos(\phi) + \gamma_{301}\cos(3\phi)\big) + \cdots$$
$$+ c_{nv,1}r^{p+v+1}\Big(\gamma_{1nv} + \gamma_{2nv}\cos(\phi) + \cdots + \gamma_{(\frac{n+v+1}{2}+2)nv}\cos((n+v+2)\phi)\Big)$$
$$+ c_{10,1}r^2\big(\gamma_{110}\sin(\phi) + \gamma_{210}\sin(3\phi)\big) + \cdots$$
$$+ c_{um,1}r^{u+m+1}\Big(\gamma_{1um}\sin(\phi) + \gamma_{2um}\sin(3\phi) + \cdots + \gamma_{(\frac{u+m+1}{2}+1)um}\sin((u+m+2)\phi)\Big)$$
$$+ c_{11,1}r^3\big(\gamma_{111} + \gamma_{211}\cos(2\phi) + \gamma_{311}\cos(4\phi)\big) + \cdots$$
$$+ c_{uv,1}r^{u+v+1}\Big(\gamma_{1uv} + \gamma_{2uv}\cos(2\phi) + \gamma_{3um}\cos(4\phi) + \cdots + \gamma_{(\frac{u+v}{2}+2)uv}\cos((u+v+2)\phi)\Big)$$
$$+ d_{00,1}r^2\big(\delta_{100} + \delta_{200}\cos(\phi) + \delta_{300}\cos(3\phi)\big) + \cdots$$
$$+ d_{nm,1}r^{n+m+2}\Big(\delta_{1nm} + \delta_{1nm}\cos(\phi) + \delta_{3nm}\cos(3\phi) + \cdots + \delta_{(\frac{n+m}{2}+3)nm}\cos((n+m+3)\phi)\Big)$$
$$+ d_{10,1}r^3\big(\delta_{110} + \delta_{210}\cos(2\phi) + \delta_{310}\cos(4\phi)\big) + \cdots$$

$$
\begin{aligned}
&+ d_{um,1} r^{u+m+2}\Big(\delta_{1um} + \delta_{2um}\cos(2\phi) + \delta_{3um}\cos(4\phi) + \cdots + \delta_{(\frac{u+m+1}{2}+2)um}\cos((u+m+3)\phi)\Big)\\
&+ d_{11,1} r^{4}\big(\delta_{111}\sin(\phi) + \delta_{211}\sin(3\phi) + \delta_{311}\sin(5\phi)\big) + \cdots\\
&+ d_{uv,1} r^{u+v+2}\Big(\delta_{1uv}\sin(\phi) + \delta_{2uv}\sin(3\phi) + \cdots + \delta_{(\frac{u+v}{2}+2)uv}\sin((u+v+3)\phi)\Big)\\
&+ e_{01,1} r^{4}\big(\varsigma_{101} + \varsigma_{201}\cos(\phi) + \varsigma_{301}\cos(3\phi) + \varsigma_{401}\cos(5\phi)\big) + \cdots\\
&+ e_{nv,1} r^{n+v+3}\Big(\varsigma_{1nv} + \varsigma_{2nv}\cos(3\phi) + \cdots + \varsigma_{(\frac{n+v+1}{2}+3)nv}\cos((n+v+4)\phi)\Big)\\
&+ e_{10,1} r^{4}\big(\varsigma_{110}\sin(\phi)\varsigma_{210}\sin(3\phi) + \varsigma_{310}\sin(5\phi)\big) + \cdots\\
&+ e_{um,1} r^{u+m+3}\Big(\varsigma_{1um}\sin(\phi) + \varsigma_{2um}\sin(3\phi) + \cdots + \varsigma_{(\frac{u+m+1}{2}+2)um}\sin((u+m+4)\phi)\Big)\\
&+ e_{11,1} r^{5}\big(\varsigma_{111} + \varsigma_{211}\cos(2\phi) + \varsigma_{311}\cos(4\phi)\big) + \cdots\\
&+ e_{uv,1} r^{u+v+3}\Big(\varsigma_{1uv} + \varsigma_{2uv}\cos(2\phi) + \varsigma_{3uv}\cos(4\phi) + \cdots + \varsigma_{(\frac{u+v}{2}+3)uv}\Big)\\
&+ p_{00,1} r(\lambda_{100} + \lambda_{200}\cos(2\phi)) + \cdots\\
&+ p_{nm,1} r^{n+m+1}\Big(\lambda_{1nm} + \lambda_{2nm}\cos(2\phi) + \cdots + \lambda_{(\frac{n+m+2}{2}+1)nm}\cos((n+m+2)\phi)\Big) + \cdots\\
&+ p_{01,1} r^{2}\big(\lambda_{101}\sin(\phi) + \lambda_{201}\sin(3\phi)\big) + \cdots\\
&+ p_{nv,1} r^{n+v+1}\Big(\lambda_{1nv}\sin(\phi) + \lambda_{2nv}\sin(3\phi) + \lambda_{3nv}\sin(5\phi) + \cdots + \lambda_{(\frac{n+v+3}{2})nv}\sin((n+v+2)\phi)\Big)\\
&+ p_{10,1} r^{2}\big(\lambda_{110} + \lambda_{210}\cos(\phi) + \lambda_{310}\cos(3\phi)\big) + \cdots\\
&+ p_{um,1} r^{u+m+1}\Big(\lambda_{1um} + \lambda_{2um}\cos(\phi) + \lambda_{3um}\cos(3\phi) + \cdots + \lambda_{(\frac{u+m+1}{2}+2)um}\cos((u+m+2)\phi)\Big),
\end{aligned}
$$

where n and m are the biggest even positive integers, u and v are the biggest odd positive integers such that

- $n+m$ smaller than or equal to m_1, m_2, n_1 and n_3,
- $n+v$ smaller than or equal to m_1, m_2, n_2 and n_4,
- $u+v$ smaller than or equal to m_1, n_1, n_2, n_3 and n_4,
- $u+m$ smaller than or equal to m_2, n_1, n_2, n_3 and n_4,

and α_{ijd}, β_{ijd}, γ_{ijd}, δ_{ijd}, ς_{ijd} and λ_{ijd} are the real constants shown during the calculation of $\int_0^\phi f_1(r,t)\,dt$ for all i, j and d.

Secondly, we have

$$
\begin{aligned}
f_2(r,\phi) = &\sum_{i+j=0}^{m_1} b_{ij,2} r^{i+j}\cos^{i+1}(\phi)\sin^{j}(\phi) + \sum_{i+j=0}^{m_2} p_{ij,2} r^{i+j+1}\cos^{i+1}(\phi)\sin^{j+1}(\phi)\\
&- \sum_{i+j=0}^{n_1} a_{ij,2} r^{i+j}\cos^{i}(\phi)\sin^{j+1}(\phi) - \sum_{i+j=0}^{n_2} c_{ij,2} r^{i+j+1}\cos^{i}(\phi)\sin^{j+2}(\phi)\\
&- \sum_{i+j=0}^{n_3} d_{ij,2} r^{i+j+2}\cos^{i}(\phi)\sin^{j+3}(\phi) - \sum_{i+j=0}^{n_4} e_{ij,2} r^{i+j+3}\cos^{i}(\phi)\sin^{j+4}(\phi)\\
&+ \sum_{\substack{i_1+j_1=0\\ i_1 \text{ even or } j_1 \text{ odd}}}^{m_1} \sum_{\substack{i_2+j_2=0\\ i_2 \text{ even or } j_2 \text{ odd}}}^{m_1} b_{i_1j_1,1} b_{i_2j_2,1} r^{i_1+j_1+i_2+j_2-1}\cos^{i_1+i_2+1}(\phi)\sin^{j_1+j_2+1}(\phi)\\
&+ \sum_{\substack{i_1+j_1=0\\ i_1 \text{ even or } j_1 \text{ odd}}}^{m_1} \sum_{\substack{i_2+j_2=0\\ i_2 \text{ even or } j_2 \text{ even}}}^{m_2} b_{i_1j_1,1} p_{i_2j_2,1} r^{i_1+j_1+i_2+j_2}\cos^{i_1+i_2+1}(\phi)\sin^{j_1+j_2+2}(\phi)
\end{aligned}
$$

$$+ \sum_{\substack{i_1+j_1=0\\ i_1 \text{ even or } j_1 \text{ odd}}}^{m_1} \sum_{\substack{i_2+j_2=0\\ i_2 \text{ odd or } j_2 \text{ even}}}^{n_1} b_{i_1j_1,1}a_{i_2j_2,1}r^{i_1+j_1+i_2+j_2-1}\cos^{i_1+i_2+2}(\phi)\sin^{j_1+j_2}(\phi)$$

$$+ \sum_{\substack{i_1+j_1=0\\ i_1 \text{ even or } j_1 \text{ odd}}}^{m_1} \sum_{\substack{i_2+j_2=0\\ i_2 \text{ odd or } j_2 \text{ odd}}}^{n_2} b_{i_1j_1,1}c_{i_2j_2,1}r^{i_1+j_1+i_2+j_2}\cos^{i_1+i_2+2}(\phi)\sin^{j_1+j_2+1}(\phi)$$

$$+ \sum_{\substack{i_1+j_1=0\\ i_1 \text{ even or } j_1 \text{ odd}}}^{m_1} \sum_{\substack{i_2+j_2=0\\ i_2 \text{ odd or } j_2 \text{ even}}}^{n_3} b_{i_1j_1,1}d_{i_2j_2,1}r^{i_1+j_1+i_2+j_2+1}\cos^{i_1+i_2+2}(\phi)\sin^{j_1+j_2+2}(\phi)$$

$$+ \sum_{\substack{i_1+j_1=0\\ i_1 \text{ even or } j_1 \text{ odd}}}^{m_1} \sum_{\substack{i_2+j_2=0\\ i_2 \text{ odd or } j_2 \text{ odd}}}^{n_4} b_{i_1j_1,1}e_{i_2j_2,1}r^{i_1+j_1+i_2+j_2+2}\cos^{i_1+i_2+2}(\phi)\sin^{j_1+j_2+3}(\phi)$$

$$+ \sum_{\substack{i_1+j_1=0\\ i_1 \text{ even or } j_1 \text{ even}}}^{m_2} \sum_{\substack{i_2+j_2=0\\ i_2 \text{ even or } j_2 \text{ even}}}^{m_2} p_{i_1j_1,1}p_{i_2j_2,1}r^{i_1+j_1+i_2+j_2+1}\cos^{i_1+i_2+1}(\phi)\sin^{j_1+j_2+3}(\phi)$$

$$+ \sum_{\substack{i_1+j_1=0\\ i_1 \text{ even or } j_1 \text{ even}}}^{m_2} \sum_{\substack{i_2+j_2=0\\ i_2 \text{ even or } j_2 \text{ odd}}}^{m_1} p_{i_1j_1,1}b_{i_2j_2,1}r^{i_1+j_1+i_2+j_2}\cos^{i_1+i_2+1}(\phi)\sin^{j_1+j_2+2}(\phi)$$

$$+ \sum_{\substack{i_1+j_1=0\\ i_1 \text{ even or } j_1 \text{ even}}}^{m_2} \sum_{\substack{i_2+j_2=0\\ i_2 \text{ odd or } j_2 \text{ even}}}^{n_1} p_{i_1j_1,1}a_{i_2j_2,1}r^{i_1+j_1+i_2+j_2}\cos^{i_1+i_2+2}(\phi)\sin^{j_1+j_2+1}(\phi)$$

$$+ \sum_{\substack{i_1+j_1=0\\ i_1 \text{ even or } j_1 \text{ even}}}^{m_2} \sum_{\substack{i_2+j_2=0\\ i_2 \text{ odd or } j_2 \text{ odd}}}^{n_2} p_{i_1j_1,1}c_{i_2j_2,1}r^{i_1+j_1+i_2+j_2+1}\cos^{i_1+i_2+2}(\phi)\sin^{j_1+j_2+2}(\phi)$$

$$+ \sum_{\substack{i_1+j_1=0\\ i_1 \text{ even or } j_1 \text{ even}}}^{m_2} \sum_{\substack{i_2+j_2=0\\ i_2 \text{ odd or } j_2 \text{ even}}}^{n_3} p_{i_1j_1,1}d_{i_2j_2,1}r^{i_1+j_1+i_2+j_2+2}\cos^{i_1+i_2+2}(\phi)\sin^{j_1+j_2+3}(\phi)$$

$$+ \sum_{\substack{i_1+j_1=0\\ i_1 \text{ even or } j_1 \text{ even}}}^{m_2} \sum_{\substack{i_2+j_2=0\\ i_2 \text{ odd or } j_2 \text{ odd}}}^{n_4} p_{i_1j_1,1}e_{i_2j_2,1}r^{i_1+j_1+i_2+j_2+3}\cos^{i_1+i_2+2}(\phi)\sin^{j_1+j_2+4}(\phi)$$

$$- \sum_{\substack{i_1+j_1=0\\ i_1 \text{ odd or } j_1 \text{ even}}}^{n_1} \sum_{\substack{i_2+j_2=0\\ i_2 \text{ odd or } j_2 \text{ even}}}^{n_1} a_{i_1j_1,1}a_{i_2j_2,1}r^{i_1+j_1+i_2+j_2-1}\cos^{i_1+i_2+1}(\phi)\sin^{j_1+j_2+1}(\phi)$$

$$- \sum_{\substack{i_1+j_1=0\\ i_1 \text{ odd or } j_1 \text{ even}}}^{n_1} \sum_{\substack{i_2+j_2=0\\ i_2 \text{ odd or } j_2 \text{ odd}}}^{n_2} a_{i_1j_1,1}c_{i_2j_2,1}r^{i_1+j_1+i_2+j_2}\cos^{i_1+i_2+1}(\phi)\sin^{j_1+j_2+2}(\phi)$$

$$- \sum_{\substack{i_1+j_1=0\\ i_1 \text{ odd or } j_1 \text{ even}}}^{n_1} \sum_{\substack{i_2+j_2=0\\ i_2 \text{ odd or } j_2 \text{ even}}}^{n_3} a_{i_1j_1,1}d_{i_2j_2,1}r^{i_1+j_1+i_2+j_2+1}\cos^{i_1+i_2+1}(\phi)\sin^{j_1+j_2+3}(\phi)$$

$$- \sum_{\substack{i_1+j_1=0\\ i_1 \text{ odd or } j_1 \text{ even}}}^{n_1} \sum_{\substack{i_2+j_2=0\\ i_2 \text{ odd or } j_2 \text{ odd}}}^{n_4} a_{i_1j_1,1}e_{i_2j_2,1}r^{i_1+j_1+i_2+j_2+2}\cos^{i_1+i_2+1}(\phi)\sin^{j_1+j_2+4}(\phi)$$

$$- \sum_{\substack{i_1+j_1=0\\ i_1 \text{ odd or } j_1 \text{ even}}}^{n_1} \sum_{\substack{i_2+j_2=0\\ i_2 \text{ even or } j_2 \text{ odd}}}^{m_1} a_{i_1j_1,1}b_{i_2j_2,1}r^{i_1+j_1+i_2+j_2-1}\cos^{i_1+i_2}(\phi)\sin^{j_1+j_2+2}(\phi)$$

$$
\begin{aligned}
&-\sum_{\substack{i_1+j_1=0\\ i_1 \text{ odd or } j_1 \text{ even}}}^{n_1}\sum_{\substack{i_2+j_2=0\\ i_2 \text{ even or } j_2 \text{ even}}}^{m_2} a_{i_1j_1,1}p_{i_2j_2,1}r^{i_1+j_1+i_2+j_2}\cos^{i_1+i_2}(\phi)\sin^{j_1+j_2+3}(\phi)\\
&-\sum_{\substack{i_1+j_1=0\\ i_1 \text{ odd or } j_1 \text{ odd}}}^{n_2}\sum_{\substack{i_2+j_2=0\\ i_2 \text{ odd or } j_2 \text{ even}}}^{n_1} c_{i_1j_1,1}a_{i_2j_2,1}r^{i_1+j_1+i_2+j_2}\cos^{i_1+i_2+1}(\phi)\sin^{j_1+j_2+2}(\phi)\\
&-\sum_{\substack{i_1+j_1=0\\ i_1 \text{ odd or } j_1 \text{ odd}}}^{n_2}\sum_{\substack{i_2+j_2=0\\ i_2 \text{ odd or } j_2 \text{ odd}}}^{n_2} c_{i_1j_1,1}c_{i_2j_2,1}r^{i_1+j_1+i_2+j_2+1}\cos^{i_1+i_2+1}(\phi)\sin^{j_1+j_2+3}(\phi)\\
&-\sum_{\substack{i_1+j_1=0\\ i_1 \text{ odd or } j_1 \text{ odd}}}^{n_2}\sum_{\substack{i_2+j_2=0\\ i_2 \text{odd or } j_2 \text{ even}}}^{n_3} c_{i_1j_1,1}d_{i_2j_2,1}r^{i_1+j_1+i_2+j_2+2}\cos^{i_1+i_2+1}(\phi)\sin^{j_1+j_2+4}(\phi)\\
&-\sum_{\substack{i_1+j_1=0\\ i_1 \text{ odd or } j_1 \text{ odd}}}^{n_2}\sum_{\substack{i_2+j_2=0\\ i_2 \text{ odd or } j_2 \text{ odd}}}^{n_4} c_{i_1j_1,1}e_{i_2j_2,1}r^{i_1+j_1+i_2+j_2+3}\cos^{i_1+i_2+1}(\phi)\sin^{j_1+j_2+5}(\phi)\\
&-\sum_{\substack{i_1+j_1=0\\ i_1 \text{ odd or } j_1 \text{ odd}}}^{n_2}\sum_{\substack{i_2+j_2=0\\ i_2 \text{ even or } j_2 \text{ odd}}}^{m_1} c_{i_1j_1,1}b_{i_2j_2,1}r^{i_1+j_1+i_2+j_2}\cos^{i_1+i_2}(\phi)\sin^{j_1+j_2+3}(\phi)\\
&-\sum_{\substack{i_1+j_1=0\\ i_1 \text{ odd or } j_1 \text{ odd}}}^{n_2}\sum_{\substack{i_2+j_2=0\\ i_2 \text{ even or } j_2 \text{ even}}}^{m_2} c_{i_1j_1,1}p_{i_2j_2,1}r^{i_1+j_1+i_2+j_2+1}\cos^{i_1+i_2}(\phi)\sin^{j_1+j_2+4}(\phi)\\
&-\sum_{\substack{i_1+j_1=0\\ i_1 \text{ odd or } j_1 \text{ even}}}^{n_3}\sum_{\substack{i_2+j_2=0\\ i_2 \text{ odd or } j_2 \text{ even}}}^{n_1} d_{i_1j_1,1}a_{i_2j_2,1}r^{i_1+j_1+i_2+j_2+1}\cos^{i_1+i_2+1}(\phi)\sin^{j_1+j_2+3}(\phi)\\
&-\sum_{\substack{i_1+j_1=0\\ i_1 \text{odd or } j_1 \text{ even}}}^{n_3}\sum_{\substack{i_2+j_2=0\\ i_2 \text{ odd or } j_2 \text{ odd}}}^{n_2} d_{i_1j_1,1}c_{i_2j_2,1}r^{i_1+j_1+i_2+j_2+2}\cos^{i_1+i_2+1}(\phi)\sin^{j_1+j_2+4}(\phi)\\
&-\sum_{\substack{i_1+j_1=0\\ i_1 \text{ odd or } j_1 \text{ even}}}^{n_3}\sum_{\substack{i_2+j_2=0\\ i_2 \text{ odd or } j_2 \text{ even}}}^{n_3} d_{i_1j_1,1}d_{i_2j_2,1}r^{i_1+j_1+i_2+j_2+3}\cos^{i_1+i_2+1}(\phi)\sin^{j_1+j_2+5}(\phi)\\
&-\sum_{\substack{i_1+j_1=0\\ i_1 \text{ odd or } j_1 \text{ even}}}^{n_3}\sum_{\substack{i_2+j_2=0\\ i_2 \text{ odd or } j_2 \text{ odd}}}^{n_4} d_{i_1j_1,1}e_{i_2j_2,1}r^{i_1+j_1+i_2+j_2+4}\cos^{i_1+i_2+1}(\phi)\sin^{j_1+j_2+6}(\phi)\\
&-\sum_{\substack{i_1+j_1=0\\ i_1 \text{ odd or } j_1 \text{ even}}}^{n_3}\sum_{\substack{i_2+j_2=0\\ i_2 \text{ odd or } j_2 \text{ odd}}}^{m_1} d_{i_1j_1,1}e_{i_2j_2,1}r^{i_1+j_1+i_2+j_2+1}\cos^{i_1+i_2}(\phi)\sin^{j_1+j_2+4}(\phi)\\
&-\sum_{\substack{i_1+j_1=0\\ i_1 \text{ odd or } j_1 \text{ even}}}^{n_3}\sum_{\substack{i_2+j_2=0\\ i_2 \text{ even or } j_2 \text{ even}}}^{m_2} d_{i_1j_1,1}p_{i_2j_2,1}r^{i_1+j_1+i_2+j_2+2}\cos^{i_1+i_2}(\phi)\sin^{j_1+j_2+5}(\phi)\\
&-\sum_{\substack{i_1+j_1=0\\ i_1 \text{ odd or } j_1 \text{ odd}}}^{n_4}\sum_{\substack{i_2+j_2=0\\ i_2 \text{ odd or } j_2 \text{ even}}}^{n_1} e_{i_1j_1,1}a_{i_2j_2,1}r^{i_1+j_1+i_2+j_2+2}\cos^{i_1+i_2+1}(\phi)\sin^{j_1+j_2+4}(\phi)\\
&-\sum_{\substack{i_1+j_1=0\\ i_1 \text{ odd or } j_1 \text{ odd}}}^{n_4}\sum_{\substack{i_2+j_2=0\\ i_2 \text{ odd or } j_2 \text{ odd}}}^{n_2} e_{i_1j_1,1}c_{i_2j_2,1}r^{i_1+j_1+i_2+j_2+3}\cos^{i_1+i_2+1}(\phi)\sin^{j_1+j_2+5}(\phi)
\end{aligned}
$$

$$- \sum_{\substack{i_1+j_1=0 \\ i_1 \text{ odd or } j_1 \text{ odd}}}^{n_4} \sum_{\substack{i_2+j_2=0 \\ i_2 \text{ odd or } j_2 \text{ even}}}^{n_3} e_{i_1j_1,1} d_{i_2j_2,1} r^{i_1+j_1+i_2+j_2+4} \cos^{i_1+i_2+1}(\phi) \sin^{j_1+j_2+6}(\phi)$$

$$- \sum_{\substack{i_1+j_1=0 \\ i_1 \text{ odd or } j_1 \text{odd}}}^{n_4} \sum_{\substack{i_2+j_2=0 \\ i_2 \text{ odd or } j_2 \text{ odd}}}^{n_4} e_{i_1j_1,1} e_{i_2j_2,1} r^{i_1+j_1+i_2+j_2+5} \cos^{i_1+i_2+1}(\phi) \sin^{j_1+j_2+7}(\phi)$$

$$- \sum_{\substack{i_1+j_1=0 \\ i_1 \text{ odd or } j_1 \text{ odd}}}^{n_4} \sum_{\substack{i_2+j_2=0 \\ i_2 \text{ even or } j_2 \text{ odd}}}^{m_1} e_{i_1j_1,1} b_{i_2j_2,1} r^{i_1+j_1+i_2+j_2+2} \cos^{i_1+i_2}(\phi) \sin^{j_1+j_2+5}(\phi)$$

$$- \sum_{\substack{i_1+j_1=0 \\ i_1 \text{ odd or } j_1 \text{ odd}}}^{n_4} \sum_{\substack{i_2+j_2=0 \\ i_2 \text{ even or } j_2 \text{ even}}}^{m_2} e_{i_1j_1,1} p_{i_2j_2,1} r^{i_1+j_1+i_2+j_2+3} \cos^{i_1+i_2}(\phi) \sin^{j_1+j_2+6}(\phi).$$

Using the integrals from Appendix, we have

$$f_{20}(r) = \frac{1}{2}\Big[U_1(r^2) + r^2U_2(r^2) + r^4U_3(r^2) + r^6U_4(r^2) + r^8U_5(r^2)\Big] = 0,$$

where:

- The polynomial $U_1(r^2)$ in r^2 has at most $[\frac{E(m_1)+E(n_1)}{2}]$ positive real solutions.
- The polynomial $U_2(r^2)$ in r^2 has at most

$$\eta_1 = \max\Big\{\Big[\frac{O(m_1)-1}{2}\Big], \Big[\frac{O(n_1)-1}{2}\Big], \Big[\frac{E(n_2)}{2}\Big], E(m_1)-1, E(n_1)-1, \Big[\frac{E(m_1)+O(n_2)-1}{2}\Big], \Big[\frac{E(m_1)+E(n_3)}{2}\Big], \Big[\frac{E(n_1)+O(n_2)-1}{2}\Big], \Big[\frac{O(m_2)+E(n_1)-1}{2}\Big], \Big[\frac{O(m_2)+E(m_1)-1}{2}\Big]\Big\},$$

  positive real solutions.
- The polynomial $U_3(r^2)$ in r^2 has at most

$$\eta_2 = \max\Big\{\Big[\frac{O(n_3)-1}{2}\Big], \Big[\frac{E(n_4)}{2}\Big], \Big[\frac{E(m_2)-2}{2}\Big], \Big[\frac{E(n_1)+E(n_3)-2}{2}\Big], \Big[\frac{E(n_1)+O(n_4)-1}{2}\Big], O(n_2)-1, \Big[\frac{O(n_2)+E(n_3)-1}{2}\Big], \Big[\frac{E(m_1)+O(n_4)-1}{2}\Big], \Big[\frac{O(m_2)+O(n_2)-2}{2}\Big], \Big[\frac{O(m_2)+E(n_3)-1}{2}\Big], O(m_2)-1\Big\},$$

  positive real solutions.
- The polynomial $U_4(r^2)$ in r^2 has at most

$$\eta_3 = \max\Big\{\Big[\frac{O(n_2)+O(n_4)-2}{2}\Big], E(n_3)-1, \Big[\frac{E(n_3)+O(n_4)-1}{2}\Big], \Big[\frac{O(m_2)+O(n_4)-2}{2}\Big]\Big\},$$

  positive real solutions.
- The polynomial $U_5(r^2)$ in r^2 has at most $O(n_4)-1$ positive real solutions.

Thus, based on the second order averaging theory, it follows that system (1.1) has at most

$$\max\Big\{\Big[\frac{E(m_1)+E(n_1)}{2}\Big], O(n_4)+3, \eta_1+1, \eta_2+2, \eta_3+3\Big\}$$

limit cycles.

5 Applications

We will give some examples. Consider the following application which corresponds to Theorem 1.1.

Example 5.1. Let

$$\begin{cases} \dot{x} = -y + \varepsilon\big((1-4x) + (3x^2+y)y\big), \\ \dot{y} = x - \varepsilon\Bigg(\Big(y + \dfrac{122}{5}x^5 + x^3y^3\Big) + \Big(2 - \dfrac{23}{234111}x^4\Big)y \\ \qquad\qquad + \Big(x^2 - \dfrac{7}{44}x^3\Big)y^2 - \Big(\dfrac{1}{2} - \dfrac{1}{555}y^2 + \dfrac{1}{76866}x^4 + 5x^4y\Big)y^3\Bigg), \end{cases} \tag{5.1}$$

where $m_1 = 1$, $m_2 = 2$, $n_1 = 6$, $n_2 = 4$, $n_3 = 3$ and $n_4 = 5$. Applying the first order averaging method, we get

$$f_{10}(r) = \frac{1}{3279616}r^7 - \frac{38593}{69296856}r^5 + \frac{3}{16}r^3 - \frac{7}{2}r = 0.$$

$f_{10}(r) = 0$ has exactly three positive real zeros $r_1 = 4.452069702$, $r_2 = 20.41113566$ and $r_3 = 37.28347607$, which satisfy

$$\frac{df_{10}(r)}{dr}\Big|_{r=r_1} = 6.571902010 \neq 0, \quad \frac{df_{10}(r)}{dr}\Big|_{r=r_2} = -98.13332340 \neq 0,$$
$$\frac{df_{10}(r)}{dr}\Big|_{r=r_3} = 1130.696273 \neq 0.$$

Then system (5.1) has exactly three limit cycles bifurcating from the periodic orbits of the linear differential system with $\varepsilon = 0$ (see Figure 1).

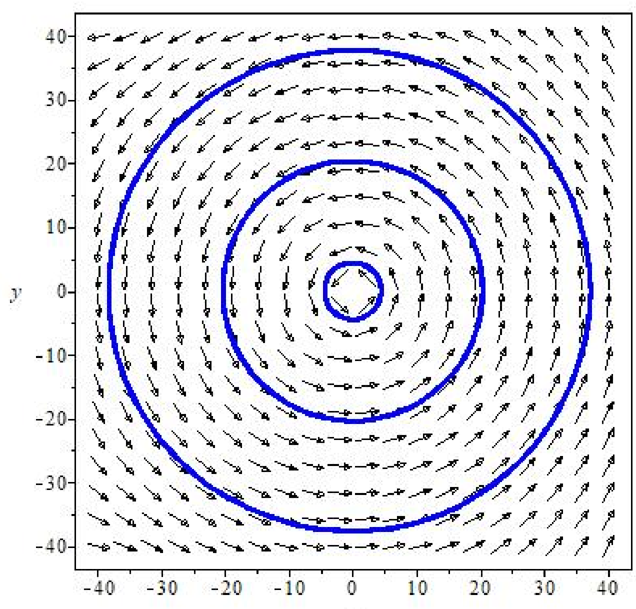

Figure 1: Three limit cycles for $\varepsilon = 10^{-12}$.

Now, we consider the application which corresponds to Theorem 1.2.

Example 5.2. Let

$$\begin{cases} \dot{x} = -y - \varepsilon\Big(\dfrac{1}{2} + 3y + (\dfrac{1}{8} - 6x^2)y\Big) + \varepsilon^2\Big(\dfrac{1}{6} + 2y - x + \Big(\dfrac{1}{5}\,x^2\Big)y\Big), \\ \dot{y} = x - \varepsilon\Big(\dfrac{181}{445}\,x + \dfrac{23}{3}\,xy + \Big(\dfrac{1}{6}\,y - yx^3\Big)y + \Big(\dfrac{34}{45}\,xy^2\Big)y^2 + \Big(\dfrac{3}{25}\,x - \dfrac{3}{25}\,y\Big)y^3\Big) \\ \qquad +\varepsilon^2\Big(\dfrac{1}{77} + \dfrac{1}{55}\,y^2 - y + \Big(\dfrac{-1}{66} + \dfrac{324}{55}\,y^2 + x^4\Big)y + \Big(\dfrac{1}{6} - x^2 y\Big)y^2 + \Big(\dfrac{-1}{87} + y + 2x\Big)y^3\Big), \end{cases} \tag{5.2}$$

where $m_1 = 1$, $m_2 = 2$, $n_1 = 2$, $n_2 = 4$, $n_3 = 3$ and $n_4 = 1$. We have $f_{10} = 0$.

To find the limit cycles, we will solve the given second order averaged equation

$$f_{20}(r) = \frac{63}{80000}\,r^7 - \frac{239}{2400}\,r^5 + \frac{2245189}{1148400}\,r^3 - \frac{61}{66}\,r = 0.$$

$f_{20}(r)$ has the following positive real roots: $r_1 = 0.6961775281$, $r_2 = 4.865327484$ and $r_3 = 10.11429439$ which satisfy

$$\frac{df_{20}(r)}{dr}\Big|_{r=r_1} = 1.802059549 \neq 0, \quad \frac{df_{20}(r)}{dr}\Big|_{r=r_2} = -67.97031102 \neq 0,$$
$$\frac{df_{20}(r)}{dr}\Big|_{r=r_3} = 1289.837920 \neq 0.$$

So, system (5.2) has exactly three limit cycles bifurcating from the periodic orbits of the linear differential system with $\varepsilon = 0$ (see Figure 2).

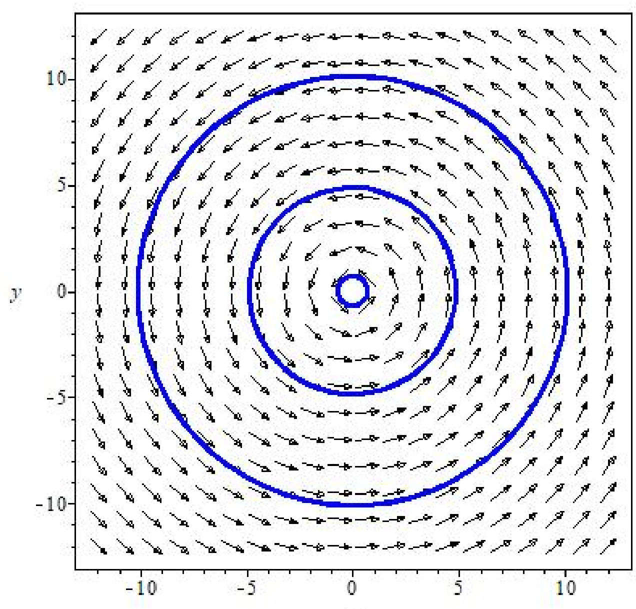

Figure 2: Three limit cycles for $\varepsilon = 10^{-9}$.

Note that for this system we have only three limit cycles. For other systems, the coefficients can be chosen in such a way that we can find exactly four limit cycles.

6 Appendix

In this section, we recall the integrals that are used in the paper. For $e \in \mathbb{N}$, we have

$$\int_0^{2\pi} \cos^{i+1}(\phi)\sin^j(\phi)\sin((2\omega+1)\phi)\,d\phi = \begin{cases} \pi A_{ij,1}^{2\omega+1} & \text{if } i=2e+1,\ j=2e+1,\ \omega=0,1,\dots,m_1,\\ \pi A_{ij,2}^{2\omega+1} & \text{if } i=2e+1,\ j=2e+1,\ \omega=0,1,\dots,m_2,\\ \pi A_{ij,3}^{2\omega+1} & \text{if } i=2e+1,\ j=2e+1,\ \omega=0,1,\dots,n_1,\\ \pi A_{ij,4}^{2\omega+1} & \text{if } i=2e+1,\ j=2e+1,\ \omega=0,1,\dots,n_2,\\ \pi A_{ij,5}^{2\omega+1} & \text{if } i=2e+1,\ j=2e+1,\ \omega=0,1,\dots,n_3,\\ \pi A_{ij,6}^{2\omega+1} & \text{if } i=2e+1,\ j=2e+1,\ \omega=0,1,\dots,n_4,\\ 0 & \text{if not,} \end{cases}$$

$$\int_0^{2\pi} \cos^{i+1}(\phi)\sin^j(\phi)\cos((2\omega)\phi)\,d\phi = \begin{cases} \pi B_{ij,1}^{2\omega} & \text{if } i=2e+1,\ j=2e,\ \omega=0,1,\dots,m_1,\\ \pi B_{ij,2}^{2\omega} & \text{if } i=2e+1,\ j=2e,\ \omega=0,1,\dots,m_2,\\ \pi B_{ij,3}^{2\omega} & \text{if } i=2e+1,\ j=2e,\ \omega=0,1,\dots,n_1,\\ \pi B_{ij,4}^{2\omega} & \text{if } i=2e+1,\ j=2e,\ \omega=0,1,\dots,n_2,\\ \pi B_{ij,5}^{2\omega} & \text{if } i=2e+1,\ j=2e,\ \omega=0,1,\dots,n_3,\\ \pi B_{ij,6}^{2\omega} & \text{if } i=2e+1,\ j=2e,\ \omega=0,1,\dots,n_4,\\ 0 & \text{if not,} \end{cases}$$

$$\int_0^{2\pi} \cos^{i+1}(\phi)\sin^j(\phi)\cos((2\omega+1)\phi)\,d\phi = \begin{cases} \pi C_{ij,1}^{2\omega+1} & \text{if } i=2e,\ j=2e,\ \omega=0,1,\dots,m_1,\\ \pi C_{ij,2}^{2\omega+1} & \text{if } i=2e,\ j=2e,\ \omega=0,1,\dots,m_2,\\ \pi C_{ij,3}^{2\omega+1} & \text{if } i=2e,\ j=2e,\ \omega=0,1,\dots,n_1,\\ \pi C_{ij,4}^{2\omega+1} & \text{if } i=2e,\ j=2e,\ \omega=0,1,\dots,n_2,\\ \pi C_{ij,5}^{2\omega+1} & \text{if } i=2e,\ j=2e,\ \omega=0,1,\dots,n_3,\\ \pi C_{ij,6}^{2\omega+1} & \text{if } i=2e,\ j=2e,\ \omega=0,1,\dots,n_4,\\ 0 & \text{if not,} \end{cases}$$

$$\int_0^{2\pi} \cos^{i}(\phi)\sin^{j+1}(\phi)\sin((2\omega+1)\phi)\,d\phi = \begin{cases} \pi D_{ij,1}^{2\omega+1} & \text{if } i=2e,\ j=2e,\ \omega=0,1,\dots,m_1,\\ \pi D_{ij,2}^{2\omega+1} & \text{if } i=2e,\ j=2e,\ \omega=0,1,\dots,m_2,\\ \pi D_{ij,3}^{2\omega+1} & \text{if } i=2e,\ j=2e,\ \omega=0,1,\dots,n_1,\\ \pi D_{ij,4}^{2\omega+1} & \text{if } i=2e,\ j=2e,\ \omega=0,1,\dots,n_2,\\ \pi D_{ij,5}^{2\omega+1} & \text{if } i=2e,\ j=2e,\ \omega=0,1,\dots,n_3,\\ \pi D_{ij,6}^{2\omega+1} & \text{if } i=2e,\ j=2e,\ \omega=0,1,\dots,n_4,\\ 0 & \text{if not,} \end{cases}$$

$$\int_0^{2\pi} \cos^{i}(\phi)\sin^{j+1}(\phi)\cos((2\omega)\phi)\,d\phi = \begin{cases} \pi E_{ij,1}^{2\omega} & \text{if } i=2e,\ j=2e+1,\ \omega=0,1,\dots,m_1,\\ \pi E_{ij,2}^{2\omega} & \text{if } i=2e,\ j=2e+1,\ \omega=0,1,\dots,m_2,\\ \pi E_{ij,3}^{2\omega} & \text{if } i=2e,\ j=2e+1,\ \omega=0,1,\dots,n_1,\\ \pi E_{ij,4}^{2\omega} & \text{if } i=2e,\ j=2e+1,\ \omega=0,1,\dots,n_2,\\ \pi E_{ij,5}^{2\omega} & \text{if } i=2e,\ j=2e+1,\ \omega=0,1,\dots,n_3,\\ \pi E_{ij,6}^{2\omega} & \text{if } i=2e,\ j=2e+1,\ \omega=0,1,\dots,n_4,\\ 0 & \text{if not,} \end{cases}$$

$$\int_0^{2\pi} \cos^i(\phi)\sin^{j+1}(\phi)\cos((2\omega+1)\phi)\,d\phi = \begin{cases} \pi F_{ij,1}^{2\omega+1} & \text{if } i=2e+1,\ j=2e+1,\ \omega=0,1,\ldots,m_1,\\ \pi F_{ij,2}^{2\omega+1} & \text{if } i=2e+1,\ j=2e+1,\ \omega=0,1,\ldots,m_2,\\ \pi F_{ij,3}^{2\omega+1} & \text{if } i=2e+1,\ j=2e+1,\ \omega=0,1,\ldots,n_1,\\ \pi F_{ij,4}^{2\omega+1} & \text{if } i=2e+1,\ j=2e+1,\ \omega=0,1,\ldots,n_2,\\ \pi F_{ij,5}^{2\omega+1} & \text{if } i=2e+1,\ j=2e+1,\ \omega=0,1,\ldots,n_3,\\ \pi F_{ij,6}^{2\omega+1} & \text{if } i=2e+1,\ j=2e+1,\ \omega=0,1,\ldots,n_4,\\ 0 & \text{if not,} \end{cases}$$

$$\int_0^{2\pi} \cos^i(\phi)\sin^{j+2}(\phi)\sin((2\omega+1)\phi)\,d\phi = \begin{cases} \pi G_{ij,1}^{2\omega+1} & \text{if } i=2e,\ j=2e+1,\ \omega=0,1,\ldots,m_1,\\ \pi G_{ij,2}^{2\omega+1} & \text{if } i=2e,\ j=2e+1,\ \omega=0,1,\ldots,m_2,\\ \pi G_{ij,3}^{2\omega+1} & \text{if } i=2e,\ j=2e+1,\ \omega=0,1,\ldots,n_1,\\ \pi G_{ij,4}^{2\omega+1} & \text{if } i=2e,\ j=2e+1,\ \omega=0,1,\ldots,n_2,\\ \pi G_{ij,5}^{2\omega+1} & \text{if } i=2e,\ j=2e+1,\ \omega=0,1,\ldots,n_3,\\ \pi G_{ij,6}^{2\omega+1} & \text{if } i=2e,\ j=2e+1,\ \omega=0,1,\ldots,n_4,\\ 0 & \text{if not,} \end{cases}$$

$$\int_0^{2\pi} \cos^i(\phi)\sin^{j+2}(\phi)\cos((2\omega)\phi)\,d\phi = \begin{cases} \pi H_{ij,1}^{2\omega} & \text{if } i=2e,\ j=2e,\ \omega=0,1,\ldots,m_1,\\ \pi H_{ij,2}^{2\omega} & \text{if } i=2e,\ j=2e,\ \omega=0,1,\ldots,m_2,\\ \pi H_{ij,3}^{2\omega} & \text{if } i=2e,\ j=2e,\ \omega=0,1,\ldots,n_1,\\ \pi H_{ij,4}^{2\omega} & \text{if } i=2e,\ j=2e,\ \omega=0,1,\ldots,n_2,\\ \pi H_{ij,5}^{2\omega} & \text{if } i=2e,\ j=2e,\ \omega=0,1,\ldots,n_3,\\ \pi H_{ij,6}^{2\omega} & \text{if } i=2e,\ j=2e,\ \omega=0,1,\ldots,n_4,\\ 0 & \text{if not,} \end{cases}$$

$$\int_0^{2\pi} \cos^i(\phi)\sin^{j+2}(\phi)\cos((2\omega+1)\phi)\,d\phi = \begin{cases} \pi I_{ij,1}^{2\omega+1} & \text{if } i=2e+1,\ j=2e,\ \omega=0,1,\ldots,m_1,\\ \pi I_{ij,2}^{2\omega+1} & \text{if } i=2e+1,\ j=2e,\ \omega=0,1,\ldots,m_2,\\ \pi I_{ij,3}^{2\omega+1} & \text{if } i=2e+1,\ j=2e,\ \omega=0,1,\ldots,n_1,\\ \pi I_{ij,4}^{2\omega+1} & \text{if } i=2e+1,\ j=2e,\ \omega=0,1,\ldots,n_2,\\ \pi I_{ij,5}^{2\omega+1} & \text{if } i=2e+1,\ j=2e,\ \omega=0,1,\ldots,n_3,\\ \pi I_{ij,6}^{2\omega+1} & \text{if } i=2e+1,\ j=2e,\ \omega=0,1,\ldots,n_4,\\ 0 & \text{if not,} \end{cases}$$

$$\int_0^{2\pi} \cos^i(\phi)\sin^{j+3}(\phi)\sin((2\omega+1)\phi)\,d\phi = \begin{cases} \pi J_{ij,1}^{2\omega+1} & \text{if } i=2e,\ j=2e,\ \omega=0,1,\ldots,m_1,\\ \pi J_{ij,2}^{2\omega+1} & \text{if } i=2e,\ j=2e,\ \omega=0,1,\ldots,m_2,\\ \pi J_{ij,3}^{2\omega+1} & \text{if } i=2e,\ j=2e,\ \omega=0,1,\ldots,n_1,\\ \pi J_{ij,4}^{2\omega+1} & \text{if } i=2e,\ j=2e,\ \omega=0,1,\ldots,n_2,\\ \pi J_{ij,5}^{2\omega+1} & \text{if } i=2e,\ j=2e,\ \omega=0,1,\ldots,n_3,\\ \pi J_{ij,6}^{2\omega+1} & \text{if } i=2e,\ j=2e,\ \omega=0,1,\ldots,n_4,\\ 0 & \text{if not,} \end{cases}$$

$$\int_0^{2\pi} \cos^i(\phi)\sin^{j+3}(\phi)\cos((2\omega)\phi)\,d\phi = \begin{cases} \pi K_{ij,1}^{2\omega} & \text{if } \ i=2e, \ \ j=2e+1, \ \ \omega=0,1,\dots,m_1, \\ \pi K_{ij,2}^{2\omega} & \text{if } \ i=2e, \ \ j=2e+1, \ \ \omega=0,1,\dots,m_2, \\ \pi K_{ij,3}^{2\omega} & \text{if } \ i=2e, \ \ j=2e+1, \ \ \omega=0,1,\dots,n_1, \\ \pi K_{ij,4}^{2\omega} & \text{if } \ i=2e, \ \ j=2e+1, \ \ \omega=0,1,\dots,n_2, \\ \pi K_{ij,5}^{2\omega} & \text{if } \ i=2e, \ \ j=2e+1, \ \ \omega=0,1,\dots,n_3, \\ \pi K_{ij,6}^{2\omega} & \text{if } \ i=2e, \ \ j=2e+1, \ \ \omega=0,1,\dots,n_4, \\ 0 & \text{if not,} \end{cases}$$

$$\int_0^{2\pi} \cos^i(\phi)\sin^{j+3}(\phi)\cos((2\omega+1)\phi)\,d\phi = \begin{cases} \pi L_{ij,1}^{2\omega+1} & \text{if } \ i=2e+1, \ \ j=2e+1, \ \ \omega=0,1,\dots,m_1, \\ \pi L_{ij,2}^{2\omega+1} & \text{if } \ i=2e+1, \ \ j=2e+1, \ \ \omega=0,1,\dots,m_2, \\ \pi L_{ij,3}^{2\omega+1} & \text{if } \ i=2e+1, \ \ j=2e+1, \ \ \omega=0,1,\dots,n_1, \\ \pi L_{ij,4}^{2\omega+1} & \text{if } \ i=2e+1, \ \ j=2e+1, \ \ \omega=0,1,\dots,n_2, \\ \pi L_{ij,5}^{2\omega+1} & \text{if } \ i=2e+1, \ \ j=2e+1, \ \ \omega=0,1,\dots,n_3, \\ \pi L_{ij,6}^{2\omega+1} & \text{if } \ i=2e+1, \ \ j=2e+1, \ \ \omega=0,1,\dots,n_4, \\ 0 & \text{if not,} \end{cases}$$

$$\int_0^{2\pi} \cos^i(\phi)\sin^{j+4}(\phi)\sin((2\omega+1)\phi)\,d\phi = \begin{cases} \pi M_{ij,1}^{2\omega+1} & \text{if } \ i=2e, \ \ j=2e+1, \ \ \omega=0,1,\dots,m_1, \\ \pi M_{ij,2}^{2\omega+1} & \text{if } \ i=2e, \ \ j=2e+1, \ \ \omega=0,1,\dots,m_2, \\ \pi M_{ij,3}^{2\omega+1} & \text{if } \ i=2e, \ \ j=2e+1, \ \ \omega=0,1,\dots,n_1, \\ \pi M_{ij,4}^{2\omega+1} & \text{if } \ i=2e, \ \ j=2e+1, \ \ \omega=0,1,\dots,n_2, \\ \pi M_{ij,5}^{2\omega+1} & \text{if } \ i=2e, \ \ j=2e+1, \ \ \omega=0,1,\dots,n_3, \\ \pi M_{ij,6}^{2\omega+1} & \text{if } \ i=2e, \ \ j=2e+1, \ \ \omega=0,1,\dots,n_4, \\ 0 & \text{if not,} \end{cases}$$

$$\int_0^{2\pi} \cos^i(\phi)\sin^{j+4}(\phi)\cos((2\omega)\phi)\,d\phi = \begin{cases} \pi N_{ij,1}^{2\omega} & \text{if } \ i=2e, \ \ j=2e, \ \ \omega=0,1,\dots,m_1, \\ \pi N_{ij,2}^{2\omega} & \text{if } \ i=2e, \ \ j=2e, \ \ \omega=0,1,\dots,m_2, \\ \pi N_{ij,3}^{2\omega} & \text{if } \ i=2e, \ \ j=2e, \ \ \omega=0,1,\dots,n_1, \\ \pi N_{ij,4}^{2\omega} & \text{if } \ i=2e, \ \ j=2e, \ \ \omega=0,1,\dots,n_2, \\ \pi N_{ij,5}^{2\omega} & \text{if } \ i=2e, \ \ j=2e, \ \ \omega=0,1,\dots,n_3, \\ \pi N_{ij,6}^{2\omega} & \text{if } \ i=2e, \ \ j=2e, \ \ \omega=0,1,\dots,n_4, \\ 0 & \text{if not,} \end{cases}$$

$$\int_0^{2\pi} \cos^i(\phi)\sin^{j+4}(\phi)\cos((2\omega+1)\phi)\,d\phi = \begin{cases} \pi P_{ij,1}^{2\omega+1} & \text{if } \ i=2e+1, \ \ j=2e, \ \ \omega=0,1,\dots,m_1, \\ \pi P_{ij,2}^{2\omega+1} & \text{if } \ i=2e+1, \ \ j=2e, \ \ \omega=0,1,\dots,m_2, \\ \pi P_{ij,3}^{2\omega+1} & \text{if } \ i=2e+1, \ \ j=2e, \ \ \omega=0,1,\dots,n_1, \\ \pi P_{ij,4}^{2\omega+1} & \text{if } \ i=2e+1, \ \ j=2e, \ \ \omega=0,1,\dots,n_2, \\ \pi P_{ij,5}^{2\omega+1} & \text{if } \ i=2e+1, \ \ j=2e, \ \ \omega=0,1,\dots,n_3, \\ \pi P_{ij,6}^{2\omega+1} & \text{if } \ i=2e+1, \ \ j=2e, \ \ \omega=0,1,\dots,n_4, \\ 0 & \text{if not,} \end{cases}$$

$$\int_0^{2\pi} \cos^{i+1}(\phi)\sin^{j+1}(\phi)\sin((2\omega+1)\phi)\,d\phi = \begin{cases} \pi Q_{ij,1}^{2\omega+1} & \text{if } i=2e+1,\ \ j=2e,\ \ \omega=0,1,\dots,m_1,\\ \pi Q_{ij,2}^{2\omega+1} & \text{if } i=2e+1,\ \ j=2e,\ \ \omega=0,1,\dots,m_2,\\ \pi Q_{ij,3}^{2\omega+1} & \text{if } i=2e+1,\ \ j=2e,\ \ \omega=0,1,\dots,n_1,\\ \pi Q_{ij,4}^{2\omega+1} & \text{if } i=2e+1,\ \ j=2e,\ \ \omega=0,1,\dots,n_2,\\ \pi Q_{ij,5}^{2\omega+1} & \text{if } i=2e+1,\ \ j=2e,\ \ \omega=0,1,\dots,n_3,\\ \pi Q_{ij,6}^{2\omega+1} & \text{if } i=2e+1,\ \ j=2e,\ \ \omega=0,1,\dots,n_4,\\ 0 & \text{if not,} \end{cases}$$

$$\int_0^{2\pi} \cos^{i+1}(\phi)\sin^{j+1}(\phi)\cos((2\omega)\phi)\,d\phi = \begin{cases} \pi R_{ij,1}^{2\omega} & \text{if } i=2e+1,\ \ j=2e+1,\ \ \omega=0,1,\dots,m_1,\\ \pi R_{ij,2}^{2\omega} & \text{if } i=2e+1,\ \ j=2e+1,\ \ \omega=0,1,\dots,m_2,\\ \pi R_{ij,3}^{2\omega} & \text{if } i=2e+1,\ \ j=2e+1,\ \ \omega=0,1,\dots,n_1,\\ \pi R_{ij,4}^{2\omega} & \text{if } i=2e+1,\ \ j=2e+1,\ \ \omega=0,1,\dots,n_2,\\ \pi R_{ij,5}^{2\omega} & \text{if } i=2e+1,\ \ j=2e+1,\ \ \omega=0,1,\dots,n_3,\\ \pi R_{ij,6}^{2\omega} & \text{if } i=2e+1,\ \ j=2e+1,\ \ \omega=0,1,\dots,n_4,\\ 0 & \text{if not,} \end{cases}$$

$$\int_0^{2\pi} \cos^{i+1}(\phi)\sin^{j+1}(\phi)\cos((2\omega+1)\phi)\,d\phi = \begin{cases} \pi S_{ij,1}^{2\omega+1} & \text{if } i=2e,\ \ j=2e+1,\ \ \omega=0,1,\dots,m_1,\\ \pi S_{ij,2}^{2\omega+1} & \text{if } i=2e,\ \ j=2e+1,\ \ \omega=0,1,\dots,m_2,\\ \pi S_{ij,3}^{2\omega+1} & \text{if } i=2e,\ \ j=2e+1,\ \ \omega=0,1,\dots,n_1,\\ \pi S_{ij,4}^{2\omega+1} & \text{if } i=2e,\ \ j=2e+1,\ \ \omega=0,1,\dots,n_2,\\ \pi S_{ij,5}^{2\omega+1} & \text{if } i=2e,\ \ j=2e+1,\ \ \omega=0,1,\dots,n_3,\\ \pi S_{ij,6}^{2\omega+1} & \text{if } i=2e,\ \ j=2e+1,\ \ \omega=0,1,\dots,n_4,\\ 0 & \text{if not,} \end{cases}$$

$$\int_0^{2\pi} \cos^{i_1+i_2+1}(\phi)\sin^{j_1+j_2+1}(\phi)\,d\phi = \begin{cases} \pi\lambda_{i_1i_2j_1j_2,1} & \text{if } i_1=2e+1,\ \ i_2=2e,\ \ j_1=2e+1,\ \ j_2=2e,\\ & \text{or } i_1=2e+1,\ \ i_2=2e,\ \ j_1=2e,\ \ j_2=2e+1,\\ & \text{or } i_1=2e,\ \ i_2=2e+1,\ \ j_1=2e+1,\ \ j_2=2e,\\ & \text{or } i_1=2e,\ \ i_2=2e+1,\ \ j_1=2e,\ \ j_2=2e+1,\\ 0 & \text{if not,} \end{cases}$$

$$\int_0^{2\pi} \cos^{i_1+i_2+2}(\phi)\sin^{j_1+j_2}(\phi)\,d\phi = \begin{cases} \pi\lambda_{i_1i_2j_1j_2,2} & \text{if } i_1=2e,\ \ i_2=2e,\ \ j_1=2e,\ \ j_2=2e,\\ & \text{or } i_1=2e,\ \ i_2=2e,\ \ j_1=2e+1,\ \ j_2=2e+1,\\ & \text{or } i_1=2e+1,\ \ i_2=2e+1,\ \ j_1=2e,\ \ j_2=2e,\\ & \text{or } i_1=2e+1,\ \ i_2=2e+1,\ \ j_1=2e+1,\ \ j_2=2e+1,\\ 0 & \text{if not,} \end{cases}$$

$$\int_0^{2\pi} \cos^{i_1+i_2+2}(\phi)\sin^{j_1+j_2+1}(\phi)\,d\phi = \begin{cases} \pi\lambda_{i_1i_2j_1j_2,3} & \text{if } i_1=2e,\ \ i_2=2e,\ \ j_1=2e,\ \ j_2=2e+1,\\ & \text{or } i_1=2e,\ \ i_2=2e,\ \ j_1=2e+1,\ \ j_2=2e,\\ & \text{or } i_1=2e+1,\ \ i_2=2e+1,\ \ j_1=2e+1,\ \ j_2=2e,\\ & \text{or } i_1=2e+1,\ \ i_2=2e+1,\ \ j_1=2e,\ \ j_2=2e+1,\\ 0 & \text{if not,} \end{cases}$$

$$\int_0^{2\pi} \cos^{i_1+i_2+2}(\phi)\sin^{j_1+j_2+2}(\phi)\,d\phi = \begin{cases} \pi\lambda_{i_1i_2j_1j_2,4} & \text{if } i_1=2e,\ \ i_2=2e,\ \ j_1=2e,\ \ j_2=2e,\\ & \text{or } i_1=2e,\ \ i_2=2e,\ \ j_1=2e+1,\ \ j_2=2e+1,\\ & \text{or } i_1=2e+1,\ \ i_2=2e+1,\ \ j_1=2e,\ \ j_2=2e,\\ & \text{or } i_1=2e+1,\ \ i_2=2e+1,\ \ j_1=2e+1,\ \ j_2=2e+1,\\ 0 & \text{if not,} \end{cases}$$

$$\int_0^{2\pi} \cos^{i_1+i_2+2}(\phi)\sin^{j_1+j_2+3}(\phi)\,d\phi = \begin{cases} \pi\lambda_{i_1i_2j_1j_2,5} & \text{if } i_1=2e,\ i_2=2e,\ j_1=2e+1,\ j_2=2e,\\ & \text{or } i_1=2e,\ i_2=2e,\ j_1=2e,\ j_2=2e+1,\\ & \text{or } i_1=2e+1,\ i_2=2e+1,\ j_1=2e+1,\ j_2=2e,\\ & \text{or } i_1=2e+1,\ i_2=2e+1,\ j_1=2e,\ j_2=2e+1,\\ 0 & \text{if not,} \end{cases}$$

$$\int_0^{2\pi} \cos^{i_1+i_2+1}(\phi)\sin^{j_1+j_2+2}(\phi)\,d\phi = \begin{cases} \pi\lambda_{i_1i_2j_1j_2,6} & \text{if } i_1=2e+1,\ i_2=2e,\ j_1=2e+1,\ j_2=2e+1,\\ & \text{or } i_1=2e+1,\ i_2=2e,\ j_1=2e,\ j_2=2e,\\ & \text{or } i_1=2e,\ i_2=2e+1,\ j_1=2e,\ j_2=2e,\\ & \text{or } i_1=2e,\ i_2=2e+1,\ j_1=2e+1,\ j_2=2e+1,\\ 0 & \text{if not,} \end{cases}$$

$$\int_0^{2\pi} \cos^{i_1+i_2+1}(\phi)\sin^{j_1+j_2+3}(\phi)\,d\phi = \begin{cases} \pi\lambda_{i_1i_2j_1j_2,7} & \text{if } i_1=2e+1,\ i_2=2e,\ j_1=2e+1,\ j_2=2e,\\ & \text{or } i_1=2e+1,\ i_2=2e,\ j_1=2e,\ j_2=2e+1,\\ & \text{or } i_1=2e,\ i_2=2e+1,\ j_1=2e+1,\ j_2=2e,\\ & \text{or } i_1=2e,\ i_2=2e+1,\ j_1=2e,\ j_2=2e+1,\\ 0 & \text{if not,} \end{cases}$$

$$\int_0^{2\pi} \cos^{i_1+i_2+1}(\phi)\sin^{j_1+j_2+4}(\phi)\,d\phi = \begin{cases} \pi\lambda_{i_1i_2j_1j_2,8} & \text{if } i_1=2e+1,\ i_2=2e,\ j_1=2e+1,\ j_2=2e+1,\\ & \text{or } i_1=2e+1,\ i_2=2e,\ j_1=2e,\ j_2=2e,\\ & \text{or } i_1=2e,\ i_2=2e+1,\ j_1=2e+1,\ j_2=2e+1,\\ & \text{or } i_1=2e,\ i_2=2e+1,\ j_1=2e,\ j_2=2e,\\ 0 & \text{if not,} \end{cases}$$

$$\int_0^{2\pi} \cos^{i_1+i_2}(\phi)\sin^{j_1+j_2+2}(\phi)\,d\phi = \begin{cases} \pi\lambda_{i_1i_2j_1j_2,9} & \text{if } i_1=2e,\ i_2=2e,\ j_1=2e,\ j_2=2e,\\ & \text{or } i_1=2e,\ i_2=2e,\ j_1=2e+1,\ j_2=2e+1,\\ & \text{or } i_1=2e+1,\ i_2=2e+1,\ j_1=2e,\ j_2=2e,\\ & \text{or } i_1=2e+1,\ i_2=2e+1,\ j_1=2e+1,\ j_2=2e+1,\\ 0 & \text{if not,} \end{cases}$$

$$\int_0^{2\pi} \cos^{i_1+i_2+1}(\phi)\sin^{j_1+j_2+5}(\phi)\,d\phi = \begin{cases} \pi\lambda_{i_1i_2j_1j_2,10} & \text{if } i_1=2e+1,\ i_2=2e,\ j_1=2e,\ j_2=2e+1,\\ & \text{or } i_1=2e+1,\ i_2=2e,\ j_1=2e+1,\ j_2=2e,\\ & \text{or } i_1=2e,\ i_2=2e+1,\ j_1=2e,\ j_2=2e+1,\\ & \text{or } i_1=2e,\ i_2=2e+1,\ j_1=2e+1,\ j_2=2e,\\ 0 & \text{if not,} \end{cases}$$

$$\int_0^{2\pi} \cos^{i_1+i_2}(\phi)\sin^{j_1+j_2+3}(\phi)\,d\phi = \begin{cases} \pi\lambda_{i_1i_2j_1j_2,11} & \text{if } i_1=2e,\ i_2=2e,\ j_1=2e,\ j_2=2e+1,\\ & \text{or } i_1=2e,\ i_2=2e,\ j_1=2e+1,\ j_2=2e,\\ & \text{or } i_1=2e+1,\ i_2=2e+1,\ j_1=2e,\ j_2=2e+1,\\ & \text{or } i_1=2e+1,\ i_2=2e+1,\ j_1=2e+1,\ j_2=2e,\\ 0 & \text{if not,} \end{cases}$$

$$\int_0^{2\pi} \cos^{i_1+i_2+1}(\phi)\sin^{j_1+j_2+6}(\phi)\,d\phi = \begin{cases} \pi\lambda_{i_1i_2j_1j_2,12} & \text{if } i_1=2e,\ i_2=2e+1,\ j_1=2e,\ j_2=2e,\\ & \text{or } i_1=2e,\ i_2=2e+1,\ j_1=2e+1,\ j_2=2e+1,\\ & \text{or } i_1=2e+1,\ i_2=2e,\ j_1=2e,\ j_2=2e,\\ & \text{or } i_1=2e+1,\ i_2=2e,\ j_1=2e+1,\ j_2=2e+1,\\ 0 & \text{if not,} \end{cases}$$

$$\int_0^{2\pi} \cos^{i_1+i_2}(\phi)\sin^{j_1+j_2+4}(\phi)\,d\phi = \begin{cases} \pi\lambda_{i_1i_2j_1j_2,13} & \text{if } i_1=2e,\ i_2=2e,\ j_1=2e,\ j_2=2e,\\ & \text{or } i_1=2e,\ i_2=2e,\ j_1=2e+1,\ j_2=2e+1,\\ & \text{or } i_1=2e+1,\ i_2=2e+1,\ j_1=2e,\ j_2=2e,\\ & \text{or } i_1=2e+1,\ i_2=2e+1,\ j_1=2e+1,\ j_2=2e+1,\\ 0 & \text{if not,} \end{cases}$$

$$\int_0^{2\pi} \cos^{i_1+i_2+1}(\phi)\sin^{j_1+j_2+7}(\phi)\,d\phi = \begin{cases} \pi\lambda_{i_1i_2j_1j_2,14} & \text{if } i_1=2e,\ i_2=2e+1,\ j_1=2e+1,\ j_2=2e,\\ & \text{or } i_1=2e,\ i_2=2e+1,\ j_1=2e,\ j_2=2e+1,\\ & \text{or } i_1=2e+1,\ i_2=2e,\ j_1=2e+1,\ j_2=2e,\\ & \text{or } i_1=2e+1,\ i_2=2e,\ j_1=2e,\ j_2=2e+1,\\ 0 & \text{if not,} \end{cases}$$

$$\int_0^{2\pi} \cos^{i_1+i_2}(\phi)\sin^{j_1+j_2+5}(\phi)\,d\phi = \begin{cases} \pi\lambda_{i_1i_2j_1j_2,15} & \text{if } i_1=2e,\ i_2=2e,\ j_1=2e,\ j_2=2e+1,\\ & \text{or } i_1=2e,\ i_2=2e,\ j_1=2e+1,\ j_2=2e,\\ & \text{or } i_1=2e+1,\ i_2=2e+1,\ j_1=2e,\ j_2=2e+1,\\ & \text{or } i_1=2e+1,\ i_2=2e+1,\ j_1=2e+1,\ j_2=2e,\\ 0 & \text{if not,} \end{cases}$$

$$\int_0^{2\pi} \cos^{i_1+i_2+2}(\phi)\sin^{j_1+j_2+4}(\phi)\,d\phi = \begin{cases} \pi\lambda_{i_1i_2j_1j_2,16} & \text{if } i_1=2e,\ i_2=2e,\ j_1=2e,\ j_2=2e,\\ & \text{or } i_1=2e,\ i_2=2e,\ j_1=2e+1,\ j_2=2e+1,\\ & \text{or } i_1=2e+1,\ i_2=2e+1,\ j_1=2e,\ j_2=2e,\\ & \text{or } i_1=2e+1,\ i_2=2e+1,\ j_1=2e+1,\ j_2=2e+1,\\ 0 & \text{if not,} \end{cases}$$

$$\int_0^{2\pi} \cos^{i_1+i_2}(\phi)\sin^{j_1+j_2+6}(\phi)\,d\phi = \begin{cases} \pi\lambda_{i_1i_2j_1j_2,17} & \text{if } i_1=2e,\ i_2=2e,\ j_1=2e,\ j_2=2e,\\ & \text{or } i_1=2e,\ i_2=2e,\ j_1=2e+1,\ j_2=2e+1,\\ & \text{or } i_1=2e+1,\ i_2=2e+1,\ j_1=2e,\ j_2=2e,\\ & \text{or } i_1=2e+1,\ i_2=2e+1,\ j_1=2e+1,\ j_2=2e+1,\\ 0 & \text{if not.} \end{cases}$$

Here, $A_{ij,h}^{2\omega+1}$, $B_{ij,h}^{2\omega}$, $C_{ij,h}^{2\omega+1}$, $D_{ij,h}^{2\omega+1}$, $E_{ij,h}^{2d}$, $F_{ij,h}^{2\omega+1}$, $G_{ij,h}^{2\omega+1}$, $H_{ij,h}^{2\omega}$, $I_{ij,h}^{2\omega+1}$, $J_{ij,h}^{2\omega+1}$, $K_{ij,h}^{2\omega}$, $L_{ij,h}^{2\omega+1}$, $M_{ij,h}^{2\omega+1}$, $N_{ij,h}^{2\omega}$, $P_{ij,h}^{2\omega+1}$, $Q_{ij,h}^{2\omega+1}$, $R_{ij,h}^{2\omega}$, $S_{ij,h}^{2\omega+1}$ for $h=1,\dots,6$ and $\lambda_{i_1i_2j_1j_2,l}$ for $l=1,\dots,17$ are the constants different from zero.

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