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MAXIMAL OPERATORS RELATED TO FEJÉR MEANS OF WALSH–FOURIER SERIES IN THE MARTINGALE HARDY SPACE $H_{1/2}$

Abstract. In this paper, we introduce and investigate a new restricted weighted maximal operator of Fejér means of Walsh–Fourier series and prove that it is bounded from the martingale Hardy space $H_{1/2}$ to the Lebesgue space $L_{1/2}$. The sharpness of this result is also established. As a consequence, we characterize the maximal subsequence of natural numbers $\{n_k : \ k \geq 0\}$ such that the restricted maximal operator of Fejér means of Walsh-Fourier series along this subsequence, defined by $\sup_{k \in \mathbb{N}} |\sigma_{n_k} F|$, is of $(H_{1/2}, L_{1/2})$ type.

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Key words and phrases. Walsh system, Fejér means, martingale Hardy space, Lebesgue spaces, maximal operators, sharpness.

რეზიუმე. ნაშრომში განვმარტავთ და გამოვიკვლევთ უოლშ-ფურიეს მწკრივების ფეიერის საშუალოების ახალ შეზღუდულ წონიან მაქსიმალურ ოპერატორს და ვამტკიცებთ, რომ იგი შემოსაზღვრულია მარტინგალური ჰარდის $H_{1/2}$ სივრციდან ლებეგის $L_{1/2}$ სივრცეში. ასევე დავადგენთ მიღებული შედეგის გაუძლიერებადობას. როგორც შედეგი, დავახასიათებთ ნატურალური რიცხვების ქვემიმდევრობებს $\{n_k : k \geq 0\}$, რომელთათვისაც უოლშ-ფურიეს მწკრივების ფეიერის საშუალოების ამ ქვემიმდევრობაზე შეზღუდული მაქსიმალური ოპერატორი $\sup_{k \in \mathbb{N}} |\sigma_{n_k} F|$ არის $(H_{1/2}, L_{1/2})$ ტიპის.

1 Introduction

In the one-dimensional case, a weak $(1,1)$-type inequality for the maximal operator σ^* of Fejér means σ_n with respect to the Walsh system defined by $\sigma^* f := \sup_{n\in\mathbb{N}} |\sigma_n f|$ was derived by Schipp in [16]. Fujii [7] and Simon [18] verified that σ^* is bounded from H_1 to L_1. Weisz [22] generalized this result and proved that the maximal operator σ^* of the Fejér means is bounded from the Hardy space $H_{1/2}$ to the space weak-$L_{1/2}$. By interpolation, it follows that σ^* is bounded from the martingale space H_p to the Lebesgue space L_p for $p > 1/2$. Goginava [8] proved that there exists a martingale $F \in H_p$ $(0 < p \leq 1/2)$ such that $\sup_{n\in\mathbb{N}} \|\sigma_n F\|_p = \infty$.

For studying the convergence of subsequences of Fejér means and their restricted maximal operators on the martingale Hardy spaces H_p for $0 < p \leq 1/2$, the central role is played by the fact that any natural number $n \in \mathbb{N}$ can be uniquely expressed as $n = \sum_{j=0}^{\infty} n_j 2^j$, $n_j \in Z_2$ $(j \in \mathbb{N})$, where only a finite number of n_j differ from zero, and their important characters $[n]$, $|n|$, $\rho(n)$ and $V(n)$ are defined respectively by

$$[n] := \min\{j \in \mathbb{N}, n_j \neq 0\}, \quad |n| := \max\{j \in \mathbb{N}, n_j \neq 0\}, \quad \rho(n) = |n| - [n],$$
$$V(n) := n_0 + \sum_{k=1}^{\infty} |n_k - n_{k-1}| \ \text{ for all } \ n \in \mathbb{N}.$$

Moreover, if $2^s \leq n_{s_1} \leq n_{s_2} \leq \cdots \leq n_{s_r} \leq 2^{s+1}$, $j = 1, \ldots r, \in \mathbb{N}$, which can be written as

$$n_{s_j} = \sum_{i=1}^{r_{s_j}} \sum_{k=l_i^{s_j}}^{t_i^{s_j}} 2^k, \ \ 0 \leq l_1^{s_j} \leq t_1^{s_j} \leq l_2^{s_j} - 2 < l_2^{s_j} \leq t_2^{s_j} \leq \cdots \leq l_{r_j}^{s_j} - 2 < l_{r_{s_j}}^{s_j} \leq t_{r_{s_j}}^{s_j},$$

we define

$$A_s := \{l_1^s, l_2^s, \ldots, l_{r_s^1}^s\} \cup \{t_1^s, t_2^s, \ldots, t_{r_s^2}^s\} = \{u_1^s, u_2^s, \ldots, u_{r_s^3}^s\}, \tag{1.1}$$

where $u_1^s < u_2^s < \cdots < u_{r_s^3}^s$ and $t_{r_{s_j}}^{s_j} = s \in A_s$ for $j = 1, 2, \ldots, r$.

The cardinality of the set A_s we denote by $|A_s|$, that is, $|A_s| := card(A_s)$. By this definition, we can conclude that $|A_s| = r_s^3 \leq r_s^1 + r_s^2$.

It is evident that $\sup_{s\in\mathbb{N}} |A_s| < \infty$ if and only if the sets $\{n_{s_1}, n_{s_2}, \ldots, n_{s_r}\}$ are uniformly finite for all $s \in \mathbb{N}_+$ and each n_{s_j} has a bounded variation $V(n_{s_j}) < c < \infty$ for each $j = 1, 2, \ldots, r$.

In [1] and [13], the maximal operator $\sigma^{*,\nabla}$, defined by

$$\sigma^{*,\nabla} F := \sup_{k\in\mathbb{N}} |\sigma_{n_k} F| \ \ (0 < p \leq 1/2), \tag{1.2}$$

has been investigated. Persson and Tephnadze [13] proved that for $0 < p < 1/2$, the maximal operator $\sigma^{*,\nabla}$, defined by (1.2), is bounded from the Hardy space H_p to the Lebesgue space L_p if and only if $\sup_{k\in\mathbb{N}} \rho(n_k) < c < \infty$.

In [1], it was proved that the maximal operator $\sigma^{*,\nabla}$, defined by (1.2) is bounded from the space $H_{1/2}$ to the Lebesgue space $L_{1/2}$ if and only if

$$\sup_{k\in\mathbb{N}} |A_{|n_k|}| \leq c < \infty. \tag{1.3}$$

In [19] it was proved that if $F \in H_{1/2}$, then there exists an absolute constant c such that

$$\|\sigma_n F\|_{H_{1/2}} \leq c V^2(n) \|F\|_{H_{1/2}}.$$

Moreover, the rate of increase of $\{V^2(n)\}$ cannot be improved. It follows that if $f \in H_{1/2}$ and $\{n_k : \ k \geq 0\}$ is any sequence of positive numbers, then $\sigma_{n_k} f$ are bounded from the Hardy space $H_{1/2}$ to the Hardy space $H_{1/2}$ if and only if $\sup_{k\in\mathbb{N}} V(n_k) < c < \infty$.

In [9], Goginava proved that the weighted maximal operator $\widetilde{\sigma}^*$, defined by

$$\widetilde{\sigma}^* f := \sup_{n\in\mathbb{N}} \frac{|\sigma_n f|}{\log^2(n+1)},$$

is bounded from the Hardy space $H_{1/2}$ to the Lebesgue space $L_{1/2}$. Moreover, it was also proved that the rate of increase of the denominator $\{\log^2(n+1)\}$ cannot be improved. Similar problem for $0<p<1/2$ was studied in [20] (see also [21]). Generalization for the two-dimensional case can be found in [11, 12] and [24]. Sharp (H_p, L_p) and $(H_p, \text{weak-}L_p)$ type inequalities of weighted maximal operators of Fejér means with sharp weights for $0<p<1/2$ have already been studied in [2] and [5].

Similar problems for the weighted maximal operators of partial sums with respect to the Walsh system can be found in [3] and [4].

In this paper, for any sequence $\{n_k : k \geq 0\}$ of positive numbers, we introduce and investigate a new restricted weighted maximal operator $\widetilde{\sigma}^{*,\nabla} F$ of Fejér means of Walsh–Fourier series and prove that it is bounded from the martingale Hardy space $H_{1/2}$ to the Lebesgue space $L_{1/2}$. We also prove that this result is sharp in a special sense. As a consequence, it is proved that the maximal operator $\sigma^{*,\nabla}$, defined by (1.2), is bounded from the Hardy space $H_{1/2}$ to the Lebesgue space $L_{1/2}$ if and only if condition (1.3) is satisfied.

2 Preliminaries

Let $\mathbb{N}_+$ denote the set of positive integers, $\mathbb{N} := \mathbb{N}_+ \cup \{0\}$. Denote by Z_2 the discrete cyclic group of order 2, that is, $Z_2 := \{0,1\}$, where the group operation is the modulo 2 addition and every subset is open. The Haar measure on Z_2 is given so that the measure of a singleton is $1/2$. Define the group G as the complete direct product of the group Z_2, with the product of the discrete topologies of Z_2. The elements of G are represented by the sequences $x := (x_0, x_1, \ldots, x_k, \ldots)$, where $x_k = 0 \vee 1$.

It is easy to give a base for the neighborhood of $x \in G$, namely, $I_0(x) := G$, $I_n(x) := \{y \in G : y_0 = x_0, \ldots, y_{n-1} = x_{n-1}\}$ $(n \in \mathbb{N})$.

Denote $I_n := I_n(0)$, $\overline{I}_n := G \setminus I_n$ and $e_n := (0, \ldots, 0, x_n = 1, 0, \ldots) \in G$, for $n \in \mathbb{N}$. Then it is not difficult to prove that

$$\overline{I}_M = \bigcup_{i=0}^{M-1} I_i \setminus I_{i+1} = \left(\bigcup_{k=0}^{M-2} \bigcup_{l=k+1}^{M-1} I_{l+1}(e_k + e_l)\right) \cup \left(\bigcup_{k=0}^{M-1} I_M(e_k)\right). \tag{2.1}$$

The norms (or quasi-norm) of the spaces L_p and weak-L_p $(0<p<\infty)$ are defined, respectively, by

$$\|f\|_p^p := \int_G |f|^p \, d\mu \quad \text{and} \quad \|f\|_{\text{weak-}L_p}^p := \sup_{\lambda>0} \lambda^p \mu\ (f > \lambda).$$

The k-th Rademacher function $r_k(x)$ is defined by $r_k(x) := (-1)^{x_k}$ $(x \in G,\ k \in \mathbb{N})$.

Now, define the Walsh system $w := (w_n : n \in \mathbb{N})$ on G as

$$w_n(x) := \prod_{k=0}^{\infty} r_k^{n_k}(x) = r_{|n|}(x)(-1)^{\sum_{k=0}^{|n|-1} n_k x_k} \quad (n \in \mathbb{N}).$$

The Walsh system is orthonormal and complete in $L_2(G)$ (see [15]).

If $f \in L_1(G)$, we can define the Fourier coefficients, partial sums of Walsh–Fourier series and Fejér means as follows:

$$\widehat{f}(n) := \int_G f w_n \, d\mu, \quad S_n f := \sum_{k=0}^{n-1} \widehat{f}(k) w_k, \quad \sigma_n f := \frac{1}{n} \sum_{k=1}^{n} S_k f \ \ (n \in \mathbb{N}_+).$$

The Dirichlet and Fejér kernels are defined by

$$D_n := \sum_{k=0}^{n-1} w_k \quad \text{and} \quad K_n := \frac{1}{n} \sum_{k=1}^{n} D_k \ \ (n \in \mathbb{N}_+).$$

Let $t, n \in \mathbb{N}$. Recall that (see [10] and [15])

$$D_{2^n}(x) = \begin{cases} 2^n, & \text{if } x \in I_n, \\ 0, & \text{if } x \notin I_n, \end{cases} \tag{2.2}$$

and

$$K_{2^n}(x) = \begin{cases} 2^{t-1}, & \text{if } x \in I_n(e_t), \ n > t, \ x \in I_t \setminus I_{t+1}, \\ \dfrac{2^n+1}{2}, & \text{if } x \in I_n, \\ 0, & \text{otherwise.} \end{cases} \tag{2.3}$$

We need the following lemmas (for details see [14] and [19]).

Lemma 2.1. *Let* $n = \sum_{i=1}^{s} \sum_{k=l_i}^{t_i} 2^k$, *where* $t_1 \geq l_1 > l_1 - 2 \geq t_2 \geq l_2 > l_2 - 2 > \cdots > t_s \geq l_s \geq 0$. *Then*

$$|nK_n| \leq c \sum_{A=1}^{s} (2^{l_A} K_{2^{l_A}} + 2^{t_A} K_{2^{t_A}} + 2^{l_A} \sum_{k=l_A}^{t_A} D_{2^k}).$$

The proof of the next lemma can be found in [1].

Lemma 2.2. *Let* $n = \sum_{i=1}^{s} \sum_{k=l_i}^{t_i} 2^k$, *where* $t_1 \geq l_1 > l_1 - 2 \geq t_2 \geq l_2 > l_2 - 2 > \cdots > t_s \geq l_s \geq 0$. *Then, for any* $i = 1, 2, \ldots, s$, *we have*

$$n|K_n(x)| \geq 2^{2t_i - 5} \ \text{ for } \ x \in E_{t_i} := I_{t_i+1}(e_{t_i+1} + e_{t_i+2}),$$
$$n|K_n(x)| \geq 2^{2l_i - 5} \ \text{ for } \ x \in E_{l_i} := I_{l_i+1}(e_{l_i-1} + e_{l_i}).$$

The σ-algebra generated by the intervals $\{I_n(x) : \ x \in G\}$ we denote by ζ_n $(n \in \mathbb{N})$. By $F = (F_n, n \in \mathbb{N})$ we denote a martingale with respect to ζ_n $(n \in \mathbb{N})$ (see, e.g., [6, 17, 22]). The maximal function F^* of a martingale F is defined by $F^* := \sup_{n \in \mathbb{N}} |F_n|$.

For $0 < p < \infty$, the Hardy martingale spaces $H_p(G)$ consist of all martingales for which $\|F\|_{H_p} := \|F^*\|_p < \infty$.

It is easy to check that for every martingale $F = (F_n, n \in \mathbb{N})$ and every $k \in \mathbb{N}$, the limit

$$\widehat{F}(k) := \lim_{n \to \infty} \int_G F_n(x) w_k(x) \, d\mu(x)$$

exists, and it is called the k-th Walsh–Fourier coefficient of F.

A bounded measurable function a is a p-atom if there exists an interval I such that

$$\operatorname{supp}(a) \subset I, \quad \int_I a \, d\mu = 0 \ \text{ and } \ |a\|_\infty \leq \mu(I)^{-1/p}.$$

Weisz [23] gave the atomic decomposition for the martingale Hardy spaces H_p, where $0 < p \leq 1$.

Lemma 2.3. *A martingale* $F = (F_n, n \in \mathbb{N})$ *is in* H_p $(0 < p \leq 1)$ *if and only if there exist a sequence* $(a_k, k \in \mathbb{N})$ *of* p*-atoms and a sequence* $(\mu_k, k \in \mathbb{N})$ *of real numbers such that for every* $n \in \mathbb{N}$,

$$\sum_{k=0}^{\infty} \mu_k S_{2^n} a_k = F_n, \quad \sum_{k=0}^{\infty} |\mu_k|^p < \infty. \tag{2.4}$$

Moreover,

$$\|F\|_{H_p} \backsim \inf \Big(\sum_{k=0}^{\infty} |\mu_k|^p \Big)^{1/p},$$

where the infimum is taken over all decompositions of F *of the form* (2.4).

Using the atomic characterization of the Hardy spaces in Lemma 2.3, one can easily prove the following useful result (see Weisz [22]).

Lemma 2.4. *Let $0 < p \leq 1$. Suppose that an operator T is σ-linear and*

$$\int_{\overline{I}} |Ta|^p \, d\mu \leq c_p < \infty$$

for every p-atom a, where I denotes the support of the atom. If T is bounded from L_∞ to L_∞, then T is also bounded from the space H_p to the space L_p.

3 The main result and its consequences

Our main result reads as follows.

Theorem 3.1.

(a) *Let $F \in H_{1/2}$ and $\{n_k : k \geq 0\}$ be a sequence of positive numbers. Then the weighted maximal operator $\widetilde{\sigma}^{*,\nabla}$, defined by*

$$\widetilde{\sigma}^{*,\nabla} F := \sup_{k\in\mathbb{N}} \frac{|\sigma_{n_k} F|}{|A_{|n_k|}|^2},$$

is bounded from the Hardy space $H_{1/2}$ to the Lebesgue space $L_{1/2}$.

(b) *(Sharpness) Let*

$$\sup_{k\in\mathbb{N}} |A_{|n_k|}| = \infty \tag{3.1}$$

and $\{\varphi_n\}$ be a nondecreasing sequence satisfying the condition

$$\overline{\lim_{k\to\infty}} \frac{|A_{|n_k|}|^2}{\varphi_{|n_k|}} = \infty. \tag{3.2}$$

Then there exists a martingale $F \in H_{1/2}$ such that the maximal operator, defined by $\sup_{k\in\mathbb{N}} \frac{|\sigma_{n_k} F|}{\varphi_{|n_k|}}$, is not bounded from the Hardy space $H_{1/2}$ to the Lebesgue space $L_{1/2}$.

Remark 3.1. For the proof of part (a), we combine Lemma 2.4 with Lemma 2.1, where the Fejér kernels are estimated by a sum of the 2^n-th Walsh–Dirichlet and Walsh–Fejér kernels, written in terms of $II^1_{l^s_A}$, $II^1_{t^s_A}$, and $II^2_{l^s_A}$. Using the classical identities for D_{2^n} and K_{2^n} given in (2.2) and (2.3), and applying Lemma 2.4, we obtain the boundedness of the maximal operator on all p-atoms.

To prove part (b), we apply the atomic decomposition of H_p spaces (see Lemma 2.3) together with the lower estimates of the Walsh–Fejér kernels (see Lemma 2.2) to show that the obtained rate of weights is sharp in the sense explained in the sharpness part.

Proof. In view of Lemma 2.4, the proof of part (a) will be complete if we prove that

$$\int_{\overline{I}_M} \left(\sup_{k\in\mathbb{N}} \frac{|\sigma_{n_{s_k}} a(x)|}{|A_{|n_{s_k}|}|^2} \right)^{1/2} d\mu(x) \leq c < \infty \quad \text{for every } 1/2\text{-atom } a. \tag{3.3}$$

We may assume that a is an arbitrary $1/2$-atom with support I, where $\mu(I) = 2^{-M}$ and $I = I_M$. Since $\sigma_n a = 0$ for $n < 2^M$, we may suppose that $n_{s_k} \geq 2^M$.

Let $x \in I_M$ and $2^s \leq n_{s_k} < 2^{s+1}$ for some $n_{s_k} \geq 2^M$. According to $\|a\|_\infty \leq 2^{2M}$ and $|n_{s_k}| = s$, using Lemma 2.1, we get

$$\begin{aligned}\frac{|\sigma_{n_{s_k}} a(x)|}{|A_{|n_{s_k}|}|^2} &\leq \frac{2^{2M}}{|A_s|^2} \int_{I_M} |K_{n_{s_k}}(x+t)|\, d\mu(t)\\ &\leq \frac{2^M}{|A_s|^2 2^s}\Big(2^M \sum_{A=1}^{r_s^1} \int_{I_M} 2^{l_A^s} |K_{2^{l_A^s}}(x+t)|\, d\mu(t)\Big)\\ &\quad + \frac{2^M}{|A_s|^2 2^s}\Big(2^M \sum_{A=1}^{r_s^2} \int_{I_M} 2^{t_A^s} |K_{2^{t_A^s}}(x+t)|\, d\mu(t)\Big)\\ &\quad + \frac{2^M}{|A_s|^2 2^s}\Big(2^M \sum_{A=1}^{r_s^1} \int_{I_M} 2^{l_j^s} \sum_{k=l_j^s}^{\infty} D_{2^k}(x+t)\, d\mu(t)\Big). \end{aligned} \tag{3.4}$$

Defining

$$\begin{aligned} II^1_{\alpha_A^s}(x) &:= 2^M \int_{I_M} 2^{\alpha_A^s} |K_{2^{\alpha_A^s}}(x+t)|\, d\mu(t), \quad \alpha = l, \quad \text{or} \quad \alpha = t,\\ II^2_{l_A^s}(x) &:= 2^M \int_{I_M} 2^{l_A^s} \sum_{k=l_A^s}^{\infty} D_{2^k}(x+t)\, d\mu(t), \end{aligned}$$

from (3.4) we can conclude that

$$\sup_{2^s \leq n_{s_k} < 2^{s+1}} \frac{|\sigma_{n_{s_k}} a|}{|A_{|n_{s_k}|}|^2} \leq \frac{2^M}{|A_s|^2 2^s}\Big(\sum_{A=1}^{r_s^1} II^1_{l_A^s} + \sum_{A=1}^{r_s^2} II^1_{t_A^s} + \sum_{A=1}^{r_s^1} II^2_{l_A^s}\Big).$$

Hence,

$$\begin{aligned} &\int_{\overline{I}_M} \Big(\sup_{2^s \leq n_{s_k} < 2^{s+1}} \frac{|\sigma_{n_{s_k}} a|}{|A_{|n_{s_k}|}|^2}\Big)^{1/2} d\mu\\ &\quad \leq \frac{2^{M/2}}{2^{s/2}|A_s|}\Big(\sum_{A=1}^{r_s^1} \int_{\overline{I}_M} |II^1_{l_A^s}|^{1/2}\, d\mu + \sum_{A=1}^{r_s^2} \int_{\overline{I}_M} |II^1_{t_A^s}|^{1/2}\, d\mu + \sum_{A=1}^{r_s^1} \int_{\overline{I}_M} |II^2_{l_A^s}|^{1/2}\, d\mu\Big). \end{aligned} \tag{3.5}$$

Since

$$\sup_{s\in\mathbb{N}} r_s \leq |A_s| \quad \text{and} \quad \sup_{s\in\mathbb{N}} r_s^2 \leq |A_s|,$$

we obtain that (3.3) holds, so Theorem 3.1 is proved if we can prove that

$$\int_{\overline{I}_M} |II^2_{l_A^s}(x)|^{1/2}\, d\mu(x) \leq c < \infty, \quad A = 1, \ldots, r_s^1, \tag{3.6}$$

and

$$\int_{\overline{I}_M} |II^1_{\alpha_A^s}(x)|^{1/2}\, d\mu(x) \leq c < \infty \tag{3.7}$$

for all $\alpha_A^s = l_A^s$, $A = 1, \ldots, r_s^1$ and $\alpha_A^s = t_A^s$, $A = 1, \ldots, r_s^2$.

Indeed, if (3.6) and (3.7) hold, from (3.5) we get

$$\int_{\overline{I}_M}\Big(\sup_{n_{s_k}\geq 2^M}\frac{|\sigma_{n_{s_k}}a|}{|A_{|n_{s_k}|}|^2}\Big)^{1/2}d\mu\leq\sum_{s=M}^{\infty}\int_{\overline{I}_M}\Big(\sup_{2^s\leq n_{s_k}<2^{s+1}}\frac{|\sigma_{n_{s_k}}a|}{|A_{|n_{s_k}|}|^2}\Big)^{1/2}$$
$$\leq\sum_{s=M}^{\infty}\frac{c2^{M/2}}{2^{s/2}}\frac{1}{A_s}\left(2r_s^1+r_s^2\right)\leq\sum_{s=M}^{\infty}\frac{c2^{M/2}}{2^{s/2}}<C<\infty.$$

It remains to prove (3.6) and (3.7).

Let $t\in I_M$ and $x\in I_{l+1}(e_k+e_l)$. If $0\leq k<l<\alpha_A^s\leq M$ or $0\leq k<l<M<\alpha_A^s$, then $x+t\in I_{l+1}(e_k+e_l)$ and, applying (2.3), we obtain

$$K_{2^{\alpha_A^s}}(x+t)=0\ \text{ and }\ II^1_{\alpha_A^s}(x)=0. \tag{3.8}$$

Let $0\leq k<\alpha_A^s\leq l<M$. Then $x+t\in I_{l+1}(e_k+e_l)$, and using (2.3), we get $2^{\alpha_A^s}|K_{2^{\alpha_A^s}}(x+t)|\leq 2^{\alpha_A^s+k}$, so that

$$II^1_{\alpha_A^s}(x)\leq 2^{\alpha_A^s+k}. \tag{3.9}$$

Analogously to (3.9), we can prove that if $0\leq\alpha_A^s\leq k<l<M$, then $2^{\alpha_A^s}|K_{2^{\alpha_A^s}}(x+t)|\leq 2^{2\alpha_A^s}$, $t\in I_M$, $x\in I_{l+1}(e_k+e_l)$, and so

$$II^1_{\alpha_A^s}(x)\leq 2^{2\alpha_A^s},\ \ t\in I_M,\ \ x\in I_{l+1}(e_k+e_l). \tag{3.10}$$

Let $t\in I_M$ and $x\in I_M(e_k)$. For $0\leq k<\alpha_A^s\leq M$ or $0\leq k<M\leq\alpha_A^s$, since $x+t\in x\in I_M(e_k)$, using (2.3) again, we find that $2^{\alpha_A^s}|K_{2^{\alpha_A^s}}(x+t)|\leq 2^{\alpha_A^s+k}$, so that

$$II^1_{\alpha_A^s}(x)\leq 2^{\alpha_A^s+k}. \tag{3.11}$$

Let $0\leq\alpha_A^s\leq k<M$. Since $x+t\in x\in I_M(e_k)$, in view of (2.3), we have $2^{\alpha_A^s}|K_{2^{\alpha_A^s}}(x+t)|\leq 2^{2\alpha_A^s}$, so that

$$II^1_{\alpha_A^s}(x)\leq 2^{2\alpha_A^s}. \tag{3.12}$$

Let $0\leq\alpha_A^s<M, A=1,\dots,s$. Combining (2.1) with (3.8)–(3.12), we get

$$\int_{\overline{I}_M}|II^1_{\alpha_A^s}|^{1/2}d\mu=\sum_{k=0}^{M-2}\sum_{l=k+1}^{M-1}\int_{I_{l+1}(e_k+e_l)}|II^1_{\alpha_A^s}|^{1/2}d\mu+\sum_{k=0}^{M-1}\int_{I_M(e_k)}|II^1_{\alpha_A^s}|^{1/2}d\mu$$
$$\leq c\sum_{k=0}^{\alpha_A^s-1}\sum_{l=\alpha_A^s}^{M-1}\int_{I_{l+1}(e_k+e_l)}2^{(\alpha_A^s+k)/2}d\mu+c\sum_{k=\alpha_A^s}^{M-2}\sum_{l=k+1}^{M-1}\int_{I_{l+1}(e_k+e_l)}2^{\alpha_A^s}d\mu$$
$$+c\sum_{k=0}^{\alpha_A^s-1}\int_{I_M(e_k)}2^{(\alpha_A^s+k)/2}d\mu+c\sum_{k=\alpha_A^s}^{M-1}\int_{I_M(e_k)}2^{\alpha_A^s}d\mu$$
$$\leq c\sum_{k=0}^{\alpha_A^s-1}\sum_{l=\alpha_A^s+1}^{M-1}\frac{2^{(\alpha_A^s+k)/2}}{2^l}+c\sum_{k=\alpha_A^s}^{M-2}\sum_{l=k+1}^{M-1}\frac{2^{\alpha_A^s}}{2^l}+c\sum_{k=0}^{\alpha_A^s-1}\frac{2^{(\alpha_A^s+k)/2}}{2^M}+c\sum_{k=\alpha_A^s}^{M-1}\frac{2^{\alpha_A^s}}{2^M}\leq C<\infty,$$

which means that (3.7) holds for $\alpha_A^s<M$. Analogously, we can prove that (3.7) holds also for $\alpha_A^s\geq M$. Hence, (3.7) holds and it remains to prove (3.6).

Next, we prove the boundedness of $II^2_{l_A^s}$. Let $t\in I_M$ and $x\in I_i\setminus I_{i+1}$. If $i\leq l_A^s-1$, since $x+t\in I_i\setminus I_{i+1}$, using (2.2), we can conclude that

$$II^2_{l_A^s}(x)=0. \tag{3.13}$$

If $l_A^s \leq i < M$, then using (2.2) again, we obtain

$$II_{l_A^s}^2(x) \leq 2^M \int_{I_M} 2^{l_A^s} \sum_{k=l_A^s}^{i} D_{2^k}(x+t)\,d\mu(t) \leq c2^{l_A^s+i}. \tag{3.14}$$

Hence, for $0 \leq l_A^s < M$, combining (2.1), (3.13) and (3.14), we get

$$\begin{aligned}\int_{\overline{I}_M} |II_{l_A^s}^2|^{1/2}\,d\mu &= \Big(\sum_{i=0}^{l_A^s-1} + \sum_{i=l_A^s+1}^{M-1}\Big) \int_{I_i \setminus I_{i+1}} |II_{l_A^s}^2|^{1/2}\,d\mu \\ &\leq c \sum_{i=l_A^s}^{M-1} \int_{I_i \setminus I_{i+1}} 2^{(l_A^s+i)/2}\,d\mu \leq c \sum_{i=l_A^s}^{M-1} 2^{(l_A^s+i)/2} \frac{1}{2^i} \leq C < \infty. \end{aligned} \tag{3.15}$$

If $M \leq l_A^s$, then $i < M \leq l_A^s$, and applying (3.13), we find that

$$\int_{\overline{I}_M} |II_{l_A^s}^2(x)|^{1/2}\,d\mu(x) = 0 \tag{3.16}$$

and (3.6) holds for this case, too, so (3.6) is proved by combining (3.15) and (3.16). Thus, the proof of part (a) is complete and we turn to the proof of part (b).

Under condition (3.2), there exists an increasing sequence $\{\alpha_k : \ k \geq 0\}$ of positive integers such that

$$\lim_{k\to\infty} \frac{|A_{|\alpha_k|}|^{1/2}}{\varphi_{|\alpha_k|}^{1/4}} = \infty$$

and

$$\sum_{k=1}^{\infty} \frac{\varphi_{|\alpha_k|}^{1/4}}{|A_{|\alpha_k|}|^{1/2}} \leq c < \infty. \tag{3.17}$$

Let

$$F_A := \sum_{\{k;|\alpha_k|<A\}} \lambda_k a_k, \quad \lambda_k := \frac{\varphi_{|\alpha_k|}^{1/2}}{|A_{|\alpha_k|}|}, \quad a_k := 2^{|\alpha_k|}(D_{2^{|\alpha_k|+1}} - D_{2^{|\alpha_k|}}).$$

Applying Lemma 2.3 and (3.17), we can conclude that $F \in H_{1/2}$. Moreover,

$$\widehat{F}(j) = \begin{cases} \dfrac{2^{|\alpha_k|}\varphi_{|\alpha_k|}^{1/2}}{|A_{|\alpha_k|}|}, & j \in \{2^{|\alpha_k|}, \ldots, 2^{|\alpha_k|+1}-1\}, \quad k \in \mathbb{N}, \\ 0, & j \notin \bigcup_{k=0}^{\infty} \{2^{|\alpha_k|}, \ldots, 2^{|\alpha_k|+1}-1\}. \end{cases} \tag{3.18}$$

Let $2^{|\alpha_k|} < j < \alpha_k$. Using (3.18), we get

$$S_j F = S_{2^{|\alpha_k|}} F + \sum_{v=2^{|\alpha_k|}}^{j-1} \widehat{F}(v) w_v = S_{2^{|\alpha_k|}} F + \frac{(D_j - D_{2^{|\alpha_k|}})2^{|\alpha_k|}}{|A_{|\alpha_k|}|}. \tag{3.19}$$

Let $2^{|\alpha_k|} \leq \alpha_{s_n} \leq 2^{|\alpha_k|+1}$. Then, according to (3.19), we have

$$\begin{aligned}\sigma_{\alpha_{s_n}} F &= \frac{1}{\alpha_{s_n}} \sum_{j=1}^{2^{|\alpha_k|}} S_j F + \frac{1}{\alpha_{s_k}} \sum_{j=2^{|\alpha_k|}+1}^{\alpha_{s_n}} S_j F \\ &= \frac{\sigma_{2^{|\alpha_k|}} F}{\alpha_{s_k}} + \frac{(\alpha_{s_n} - 2^{|\alpha_k|}) S_{2^{|\alpha_k|}} F}{\alpha_{s_n}} + \frac{2^{|\alpha_k|}\varphi_{|\alpha_k|}^{1/2}}{|A_{|\alpha_k|}|\alpha_{s_n}} \sum_{j=2^{|\alpha_k|}+1}^{\alpha_{s_n}} (D_j - D_{2^{|\alpha_k|}}) \\ &:= III_1 + III_2 + III_3. \end{aligned} \tag{3.20}$$

Since $D_{j+2^m}=D_{2^m}+w_{2^m}D_j$, when $j<2^m$, we obtain

$$|III_3|=\frac{2^{|\alpha_k|}\varphi_{|\alpha_n|}^{1/2}}{|A_{|\alpha_k|}|\alpha_{s_n}}\Big|\sum_{j=1}^{\alpha_{s_n}-2^{|\alpha_k|}}D_j\Big|\geq\frac{\varphi_{|\alpha_k|}^{1/2}}{2|A_{|\alpha_k|}|}(\alpha_{s_n}-2^{|\alpha_k|})|K_{\alpha_{s_n}-2^{|\alpha_k|}}|.$$

Combining the well-known estimates (see [14])

$$\|S_{2^k}F\|_{H_{1/2}}\leq c_1\|F\|_{H_{1/2}}\ \text{ and }\ \|\sigma_{2^k}F\|_{H_{1/2}}\leq c_2\|F\|_{H_{1/2}},\ \ k\in\mathbb{N},$$

we obtain

$$\|III_1\|_{1/2}\leq C\ \text{ and }\ \|III_2\|_{1/2}\leq C. \tag{3.21}$$

Let $2^{|\alpha_k|}\leq\alpha_{s_1}\leq\alpha_{s_2}\leq\cdots\leq\alpha_{s_r}\leq 2^{|\alpha_k|+1}$ be natural numbers which generate the set

$$A_{|\alpha_k|}=\{l_1^{|\alpha_k|},l_2^{|\alpha_k|},\dots,l_{r^1_{|\alpha_k|}}^{|\alpha_k|}\}\cup\{t_1^{|\alpha_k|},t_2^{|\alpha_k|},\dots,t_{r^2_{|\alpha_k|}}^{|\alpha_k|}\},$$

where $t_1^{|\alpha_k|}\geq l_1^{|\alpha_k|}>l_1^{|\alpha_k|}-2\geq t_2^{|\alpha_k|}\geq l_2^{|\alpha_k|}>l_2^{|\alpha_k|}-2\geq\cdots\geq t_{|\alpha_k|}^{|\alpha_k|}\geq l_{|\alpha_k|}^{|\alpha_k|}\geq 0$, and for any $1\leq i\leq r^1_{|\alpha_k|}$, there exists $\alpha_{s_n}\in A_{|\alpha_k|}$, for some $1\leq n\leq r$ such that $\alpha_{s_n}=\sum_{i=1}^{r_n}\sum_{k=l_i^n}^{t_i^n}2^k$, where $l_u^{|\alpha_k|}=l_i$ for some $1\leq u\leq r^1_{|\alpha_k|}$.

Since $\mu\{E_{l_i}\}\geq 1/2^{l_i+1}$, using Lemma 2.2, we get

$$\int_{E_{l_i}}\Big(\sup_{k\in\mathbb{N}}\frac{|\sigma_{\alpha_{s_k}}F|}{\varphi_{|\alpha_{s_k}|}}\Big)^{1/2}d\mu\geq\int_{E_{l_i}}\Big(\frac{\sigma_{\alpha_{s_n}}F}{\varphi_{|\alpha_{s_n}|}}\Big)^{1/2}d\mu\geq\frac{1}{2^5|A_{|\alpha_k|}|^{1/2}\varphi_{|\alpha_k|}^{1/4}}. \tag{3.22}$$

On the other hand, for any $1\leq i\leq r^2_{|\alpha_k|}$, there exists the number α_{s_n} for some $1\leq n\leq r$, where $t_u^{|\alpha_k|}=t_i$ for some $1\leq u\leq r^2_{|\alpha_k|}$. According to the fact that $\mu\{E_{t_i}\}\geq 1/2^{t_i+3}$, using again Lemma 2.2 for some α_{s_n} and $1\leq i\leq r_s^2$, we also get

$$\int_{E_{l_i}}\Big(\sup_{k\in\mathbb{N}}\frac{|\sigma_{\alpha_{s_k}}F|}{\varphi_{|\alpha_{s_k}|}}\Big)^{1/2}d\mu\geq\int_{E_{t_i}}\Big(\frac{\sigma_{\alpha_{s_n}}F}{\varphi_{|\alpha_{s_n}|}}\Big)^{1/2}d\mu\geq\frac{1}{2^7|A_{|\alpha_k|}|^{1/2}\varphi_{|\alpha_k|}^{1/4}}. \tag{3.23}$$

Combining (3.20)–(3.23) and Lemma 2.2 for sufficiently big α_k, we obtain

$$\begin{aligned}\int_G\Big(\sup_{n\in\mathbb{N}}\frac{|\sigma_{\alpha_{s_n}}F|}{\varphi_{|\alpha_{s_n}|}}\Big)^{1/2}d\mu&\geq\|III_3\|_{1/2}^{1/2}-\|III_2\|_{1/2}^{1/2}-\|III_1\|_{1/2}^{1/2}\\ &\geq c\sum_{i=1}^{r^1_{|\alpha_k|}-1}\int_{E_{l_i}}\frac{1}{|A_{|\alpha_k|}|^{1/4}\varphi_{|\alpha_k|}^{1/8}}d\mu+c\sum_{i=1}^{r^2_{|\alpha_k|}-1}\int_{E_{t_i}}\frac{1}{|A_{|\alpha_k|}|^{1/4}\varphi_{|\alpha_k|}^{1/8}}d\mu-2C\\ &\geq\frac{c(r^1_{|\alpha_k|}+r^2_{|\alpha_k|})}{2^7|A_{|\alpha_k|}|^{1/2}\varphi_{|\alpha_k|}^{1/4}}-2C\geq\frac{c|A_{|\alpha_k|}|^{1/2}}{2^8\varphi_{|\alpha_k|}^{1/4}}\to\infty\ \text{ as }\ k\to\infty.\end{aligned}$$

Thus part (b) is also proved and the proof is complete. □

As a consequence, we get the results proved in [1].

Corollary 3.1.

(a) *Let $f\in H_{1/2}(G)$ and $\{n_k:\ k\geq 0\}$ be a sequence of positive numbers and let $\{n_{s_i}:\ 1\leq i\leq r\}\subset\{n_k:\ k\geq 0\}$ be the numbers such that $2^s\leq n_{s_1}\leq n_{s_2}\leq\cdots\leq n_{s_r}\leq 2^{s+1}$, $s\in\mathbb{N}$. If the* *sets A_s, defined by (1.1), are uniformly finite for all $s \in \mathbb{N}$, that is, the cardinality of the sets A_s is uniformly finite, i.e., $\sup_{s\in\mathbb{N}} |A_s| < c < \infty$, then the restricted maximal operator $\widetilde{\sigma}^{*,\nabla}$, defined by*

$$\widetilde{\sigma}^{*,\nabla} F = \sup_{k\in\mathbb{N}} |\sigma_{n_k} F|, \tag{3.24}$$

is bounded from the Hardy space $H_{1/2}$ to the Lebesgue space $L_{1/2}$.

(b) *(sharpness) Let $\{n_k : k \geq 0\}$ be a sequence of positive numbers such that (3.1) holds. Then there exists a martingale $f \in H_{1/2}(G)$ such that the maximal operator, defined by (3.24), is not bounded from the Hardy space $H_{1/2}$ to the Lebesgue space $L_{1/2}$.*

Corollary 3.2. *Let $F \in H_{1/2}$. Then the maximal operator $\sigma^{*,\nabla}$, defined by (1.2), is bounded from the Hardy space $H_{1/2}$ to the Lebesgue space $L_{1/2}$ if and only if condition (1.3) is fulfilled, which is equivalent to the fact that any sequence of positive numbers $\{n_k : k \geq 0\}$, satisfying the condition $n_k \in [2^s, 2^{s+1})$, is finite for every $s \in \mathbb{N}_+$ and every $\{n_k : k \geq 0\}$ has bounded variation, i.e., $\sup_{k\in\mathbb{N}} V(n_k) < c < \infty$.*

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