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# ON MILD NONNEGATIVE SOLUTIONS OF A CAPUTO ITERATIVE FRACTIONAL RELAXATION DIFFERENTIAL EQUATION: EXISTENCE, UNIQUENESS AND STABILITY

**Abstract.** This paper investigates the existence, uniqueness, continuous dependence and Ulam stability of mild nonnegative solutions for a class of semi-linear Caputo iterative fractional relaxation differential equations with nonzero initial conditions. To achieve this, we first transform the given equation into an integral equation. The existence of the mild nonnegative solution is then established by applying the Schauder fixed point theorem. Next, we analyze the uniqueness, continuous dependence and Ulam stability of the mild nonnegative solution, accompanied by illustrative examples that serve to validate and exemplify the theoretical results.

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**Key words and phrases.** Iterative fractional relaxation differential equations, mild nonnegative solutions, fixed point theorems, continuous dependence, Ulam stability.

# 1 Introduction

In recent decades, the growing complexity of real-world problems across various scientific and industrial domains has led to an increasing reliance on fractional differential equations (FDEs). These equations have emerged as powerful tools for modeling systems with memory effects and hereditary properties, making them essential in fields such as engineering, finance [5], physics [24], control theory and medical sciences [3, 20]. Unlike classical integer-order models, FDEs effectively capture nonlocal dynamics, which are essential in applications ranging from control systems and viscoelastic materials [9] to biological processes [21] and anomalous diffusion models. For foundational theories and comprehensive treatments of FDEs, we refer the reader to the research conducted by Abbas et al. [1], Benchohra et al. [6–8], Kilbas et al. [16], Miller and Ross [18] and Podlubny [19].

Later, as the theory of FDEs evolved, researchers introduced iterative fractional differential equations (IFDEs) to describe dynamic processes that evolve through nested or recursive behaviors. A representative form is

$$\psi'(y) = g\big(\psi^{[0]}(y), \psi^{[1]}(y), \psi^{[2]}(y), \dots, \psi^{[n]}(y)\big),$$

where

$$\psi^{[0]}(y) = y,\ \psi^{[1]}(y) = \psi(y),\ \psi^{[2]}(y) = \psi(\psi(y)),\ \dots,\ \psi^{[n]}(y) = \psi^{[n-1]}(\psi(y)).$$

These equations extend classical FDEs to model systems where the current state depends on multiple previous iterations, making them particularly useful for problems involving successive approximations, optimization techniques, and stepwise processes in applied mathematics. Moreover, their ability to incorporate long-term memory and feedback effects makes them particularly promising in epidemiological modeling and medical applications, where systems often evolve based on past exposures, treatments, or biological responses.

Building upon the framework introduced in [12, 14, 15, 22], we establish in this paper the existence, uniqueness, continuous dependence and the Ulam stability of the mild nonnegative solutions for the semi-linear fractional relaxation differential equation with nonzero initial conditions

$$\begin{cases} {}^{C}D^{\alpha}_{0^+}\psi(y) + \varpi\psi(y) = g(y, \psi^{[1]}(y), \psi^{[2]}(y), \dots, \psi^{[n]}(y)), & y \in K, \\ \psi(0) = \beta_1, \quad \psi'(0) = \beta_2, \end{cases} \tag{1.1}$$

where $K = [0, T]$, ${}^{C}D^{\alpha}_{0^+}$ is the fractional Caputo derivative of order $\alpha \in (1, 2)$, $g \in C(K \times \mathbb{R}^n, \mathbb{R}_+)$ which also satisfies other conditions that will be revealed later, and $\varpi$, $\beta_1$, $\beta_2$ are positive real numbers.

In order to accomplish our objective, we first transform the equation in (1.1) into an integral equation using the Laplace transform, for an in-depth information of this technique, we refer the reader to [10, 23]. Subsequently, we establish the existence of the mild nonnegative solution by applying the Schauder fixed point theorem; for a comprehensive explanation of this theorem, we direct the reader to [2]. Finally, we analyze the uniqueness, continuous dependence and Ulam stability of the mild nonnegative solution.

The remainder of this paper is organized as follows. Section 2 revisits the fundamental definitions, essential lemmas, and key notation necessary for the subsequent sections. Section 3 presents the main results concerning the existence, uniqueness, continuous dependence and Ulam stability of the mild nonnegative solution, along with illustrative examples that reinforce the theoretical findings. Finally, Section 4 concludes the paper.

# 2 Preliminaries

Let $C(K, \mathbb{R})$ be the Banach space of all continuous functions mapping the compact interval $K = [0, T]$ into $\mathbb{R}$, equipped with the supremum norm

$$\|\psi\| = \|\psi\|_\infty = \sup\big\{|\psi(y)|,\ y \in K\big\}.$$

We define for $L > 0$ and $M > 0$ the two following sets:

$$B(K, L) = \big\{\psi \in C(K, \mathbb{R}) : \quad 0 \le \psi(y) \le L \le T\big\}$$

and

$$B_M(K,L) = \Big\{ \psi \in B(K,L) : \ \ |\psi(y_2) - \psi(y_1)| \leq M|y_2 - y_1|, \ \ \forall\, y_1, y_2 \in K \Big\}.$$

Clearly, $B_M(K,L)$ forms a nonempty, closed, bounded and convex subset of $B(K,L)$.

Additionally, we assume that the positive continuous function $g$ is globally Lipschitz. This means that there exists a set of strictly positive constants $\{\kappa_i\}_{i=1}^n$ such that

$$\big|g(y,\psi_1,\psi_2,\dots,\psi_n) - g(y,\phi_1,\phi_2,\dots,\phi_n)\big| \leq \sum_{i=1}^{n} \kappa_i |\psi_i - \phi_i|. \tag{2.1}$$

We define the constants

$$\gamma = \sup_{y\in K} |g(y,0,0,\dots,0)|, \quad \rho = \gamma + L \sum_{i=1}^{n} \kappa_i \sum_{j=0}^{i-1} M^j,$$

and

$$\lambda = 1 + \varpi T^\alpha \Big( \frac{1}{\Gamma(\alpha+2)} + \frac{1}{\Gamma(\alpha+1)} \Big).$$

**Definition 2.1** ([16])**.** Let $\Omega = [0,b]$ be a finite interval, where $0 < b < \infty$, $\psi \in L^1(\Omega,\mathbb{R})$ and $\alpha > 0$, the integral

$$I_{0^+}^{\alpha} \psi(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} \psi(\tau)\, d\tau, \ \ t > 0,$$

is the left-sided Riemann–Liouville fractional integral of order $\alpha$, where $\Gamma(\alpha)$ denotes the Gamma function.

**Definition 2.2** ([16])**.** For $\psi \in C^n([0,b])$, its left-sided Caputo fractional derivative of order $\alpha$, where $0 < \alpha < n$, $n = [\alpha] + 1$, is defined as

$${}^{C}D_{0^+}^{\alpha} \psi(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\tau)^{n-\alpha-1} \psi^{(n)}(\tau)\, d\tau = I_{0^+}^{n-\alpha} \psi(t), \ \ t > 0.$$

**Definition 2.3** ([16])**.** The Laplace transform of a function $\psi$, defined for $y > 0$, is given by

$$\Psi(s) = L[\psi(y)](s) = \int_0^\infty e^{-sy} \psi(y)\, dy = \lim_{\tau\to\infty} \int_0^\tau e^{-sy} \psi(y)\, dy, \ \ s \in \mathbb{C}. \tag{2.2}$$

The fundamental properties of the direct Laplace transform are:

$$L\big[\psi_1(y) + \psi_2(y)\big](s) = L[\psi_1(y)](s) + L[\psi_2(y)](s) = \Psi_1(s) + \Psi_2(s),$$
$$L[k\psi(y)](s) = kL[\psi(y)](s) = k\Psi(s),$$
$$L\big[\psi_1(y) * \psi_2(y)\big](s) = L[\psi_1(y)](s) \times L[\psi_2(y)](s) = \Psi_1(s) \times \Psi_2(s).$$

Some direct Laplace transforms are:

$$L[c](s) = \frac{c}{s}, \ \ c \text{ is a constant},$$
$$L[e^{ay}](s) = \frac{1}{(s-a)}, \ \ \mathrm{Re}(s) > \mathrm{Re}(a),$$
$$\text{For } \ n = \{0,1,\dots\}, \ L[y^n](s) = \frac{n!}{s^{n+1}}, \ \ \mathrm{Re}(s) > 0,$$
$$L\big[t^{\beta-1} E_{\alpha,\beta}(-\lambda t^\alpha)\big](s) = \frac{s^{\alpha-\beta}}{s^\alpha + \lambda},$$

where $E_{\alpha,\beta}$ is the two-parameter Mittag–Lefler function.

**Definition 2.4.** The inverse Laplace transform of a function $\varphi$, defined for $y > 0$, is given by

$$L^{-1}[\varphi(s)](y) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{sy}\varphi(s)ds, \quad c > 0.$$

The fundamental properties of the inverse Laplace transform are:

$$L^{-1}\big[\varphi_1(s)+\varphi_2(s)\big](y) = L^{-1}[\varphi_1(s)](y) + L^{-1}[\varphi_2(s)](y),$$
$$L^{-1}[k\varphi(s)](y) = kL^{-1}[\varphi(s)](y),$$
$$L^{-1}\big[\varphi_1(s)\times\varphi_2(s)\big](y) = L^{-1}[\varphi_1(s)](y) * L^{-1}[\varphi_2(s)](y).$$

*Remark* 2.1 ([16]). For the functions $\psi$ and $\varphi$ with appropriate regularity, we have

$$L(L^{-1}\varphi) = \varphi \text{ and } L^{-1}(L\psi) = \psi.$$

**Definition 2.5** ([4])**.** The Laplace transform of the Caputo fractional derivative is

$$L\big[{}^{C}D^{\alpha}_{0^+}\psi(t)\big](s) = s^{\alpha}\Psi(s) - \sum_{k=0}^{n-1} s^{\alpha-1-k}\psi^{(k)}(0), \quad n-1 < \alpha < n.$$

**Definition 2.6.** A function $\psi \in B_M(K, L)$ is called a mild solution of problem (1.1) if it satisfies the associated integral equation derived from the original problem.

**Lemma 2.1.** *We call $\psi \in C(K,\mathbb{R})$ a mild solution of the initial value problem* (1.1) *if*

$$\psi(y) = \int_0^y (y-t)^{\alpha-1}E_{\alpha,\alpha}(-\varpi(y-t)^{\alpha})g\big(t,\psi^{[1]}(t),\psi^{[2]}(t),\ldots,\psi^{[n]}(t)\big)\,dt$$
$$+ \beta_1E_{\alpha}(-\varpi y^{\alpha}) + \beta_2 yE_{\alpha,2}(-\varpi y^{\alpha}), \quad y \in K.$$

*Proof.* Let $\psi \in C(K,\mathbb{R})$ be a mild solution to (1.1), then we have

$$ {}^{C}D^{\alpha}_{0^+}\psi(y) + \varpi\psi(y) = g\big(y,\psi^{[1]}(y),\psi^{[2]}(y),\ldots,\psi^{[n]}(y)\big). \tag{2.3}$$

Applying the direct Laplace transform to (**??**), we derive the expression

$$t^{\alpha}\Psi(t) + \varpi\Psi(t) - \beta_1t^{\alpha-1} - \beta_2t^{\alpha-2} = G(t),$$

where

$$\Psi(t) = L[\Psi(y)](t) \text{ and } G(t) = L\big[g(y,\psi^{[1]}(y),\psi^{[2]}(y),\ldots,\psi^{[n]}(y))\big](t).$$

We then obtain

$$\Psi(t) = \frac{G(t)}{t^{\alpha}+\varpi} + \beta_1\frac{t^{\alpha-1}}{t^{\alpha}+\varpi} + \beta_2\frac{t^{\alpha-2}}{t^{\alpha}+\varpi}. \tag{2.4}$$

Finally, applying the inverse Laplace transform to (2.3), we get the following integral form:

$$\psi(y) = \int_0^y (y-t)^{\alpha-1}E_{\alpha,\alpha}(-\varpi(y-t)^{\alpha})g\big(t,\psi^{[1]}(t),\psi^{[2]}(t),\ldots,\psi^{[n]}(t)\big)\,dt$$
$$+ \beta_1E_{\alpha}(-\varpi y^{\alpha}) + \beta_2 yE_{\alpha,2\alpha}(-\varpi y^{\alpha}).$$

Thus, the desired result is obtained. □

**Lemma 2.2** ([26])**.** *For $\mu, \upsilon \in B_M(K, L)$, we have*

$$\|\mu^{[m]} - \upsilon^{[m]}\| \le \sum_{j=0}^{m-1} M^j\|\mu - \upsilon\|.$$

**Theorem 2.1** (Schauder's fixed point theorem [4])**.** *Let $E$ be a Banach space, and let $C$ be a nonempty compact and convex subset of $E$. If $A : C \to C$ is a continuous mapping, then $A$ has at least one fixed point.*

**Lemma 2.3** ([13, 16, 25])**.**

(i) *Let $\alpha > 0$, $\beta < 1$ and $\varpi > 0$, and consider $y, y_1, y_2 \in K$ with $y_1 \le y_2$. We have*

$$E_\alpha(-\varpi y^\alpha) \le 1, \quad E_{\alpha,\alpha+\beta}(-\varpi y^\alpha) \le \frac{1}{\Gamma(\alpha+\beta)} \tag{2.5}$$

*and*

$$E_\alpha(-\varpi y_2^\alpha) \le E_\alpha(-\varpi y_1^\alpha), \quad E_{\alpha,\alpha+\beta}(-\varpi y_2^\alpha) \le E_{\alpha,\alpha+\beta}(-\varpi y_1^\alpha). \tag{2.6}$$

(ii)

$$\Big(\frac{d}{dy}\Big)[E_{\alpha,\beta}(y)] = E^2_{\alpha,\beta+\alpha}(y), \tag{2.7}$$

*where $E^2_{\alpha,\beta+\alpha}$ is the generalized Mittag–Leffler function.*

(iii) *For $\mathrm{Re}(\alpha) > 0$, $\mathrm{Re}(\beta) > 1$, we have*

$$\alpha E^2_{\alpha,\beta} = E_{\alpha,\beta-1} + (\beta - \alpha - 1)E_{\alpha,\beta}. \tag{2.8}$$

# 3 Main results

In this section, we present the existence result using Theorem 2.1. In addition, we formulate sufficient conditions to ensure the uniqueness of the mild solution to the initial value problem stated in (1.1). Subsequently, we address the continuous dependence on initial data and investigate the Ulam stability of the solution.

To make use of the Schauder fixed point theorem, the equation in (1.1) is reformulated as follows:

$$\psi(y) = (T\psi)(y),$$

where $T : B_M(K, L) \to C(K, \mathbb{R})$ is defined by

$$(T\psi)(y) = \int_0^y (y-t)^{\alpha-1} E_{\alpha,\alpha}\big(-\varpi(y-t)^\alpha\big) g\big(t, \psi^{[1]}(t), \psi^{[2]}(t), \dots, \psi^{[n]}(t)\big)\, dt + \beta_1 E_\alpha(-\varpi y^\alpha) + \beta_2 y E_{\alpha,2}(-\varpi y^\alpha). \tag{3.1}$$

## 3.1 Existence

Clearly, $B_M(K, L)$ is a compact set in $C(K, \mathbb{R})$. With the aim of proving that the operator $T$ has at least one fixed point, we will demonstrate that $T$ is continuous, and $T(B_M(K, L)) \subseteq B_M(K, L)$ (i.e., $\forall\, \psi \in B_M(K, L)$, $T\psi \in B_M(K, L)$).

**Lemma 3.1.** *Assume that condition (2.1) holds, then the operator $T$ is continuous.*

*Proof.* Let $\psi, \varphi \in B_M(K, L)$, we have

$$\begin{aligned}\big|(T\psi)(y) - (T\varphi)(y)\big| &= \Big| \int_0^y (y-t)^{\alpha-1} E_{\alpha,\alpha}(-\varpi(y-t)^\alpha) g\big(t, \psi^{[1]}(t), \psi^{[2]}(t), \dots, \psi^{[n]}(t)\big)\, dt \\ &\quad - \int_0^y (y-t)^{\alpha-1} E_{\alpha,\alpha}\big(-\varpi(y-t)^\alpha\big) g\big(t, \varphi^{[1]}(t), \varphi^{[2]}(t), \dots, \varphi^{[n]}(t)\big)\, dt \Big|\end{aligned}$$

$$\leq \frac{1}{\Gamma(\alpha)} \int_0^y (y-t)^{\alpha-1} \Big| g\big(t, \psi^{[1]}(t), \psi^{[2]}(t), \ldots, \psi^{[n]}(t)\big) - g\big(t, \varphi^{[1]}(t), \varphi^{[2]}(t), \ldots, \varphi^{[n]}(t)\big) \Big| \, dt.$$

Using (2.1), we obtain

$$\big|(T\psi)(y) - (T\varphi)(y)\big| \leq \frac{1}{\Gamma(\alpha)} \int_0^y (y-t)^{\alpha-1} \sum_{i=1}^{n} c_i \|\psi^{[i]} - \varphi^{[i]}\| \, dt.$$

Applying Lemma 2.2, we derive

$$\begin{aligned} &\big|(T\psi)(y) - (T\varphi)(y)\big| \\ &\quad \leq \frac{\alpha}{\Gamma(\alpha+1)} \sum_{i=1}^{n} \kappa_i \sum_{j=0}^{i-1} M^j \|\psi - \varphi\| \int_0^y (y-t)^{\alpha-1} \, dt \leq \frac{T^\alpha}{\Gamma(\alpha+1)} \sum_{i=1}^{n} \kappa_i \sum_{j=0}^{i-1} M^j \|\psi - \varphi\|. \end{aligned} \tag{3.2}$$

Then

$$\big\|(T\psi)(y) - (T\varphi)(y)\big\| \leq \frac{T^\alpha}{\Gamma(\alpha+1)} \sum_{i=1}^{n} \kappa_i \sum_{j=0}^{i-1} M^j \|\psi - \varphi\|.$$

Hence, $T$ is continuous. □

**Lemma 3.2.** *Assume that condition (2.1) holds. If*

$$\Big( \frac{\rho}{\Gamma(\alpha+1)} T^\alpha + \beta_1 + \beta_2 T \Big) \leq L \tag{3.3}$$

*and*

$$\Big( \frac{\rho}{\Gamma(\alpha)} T^{\alpha-1} + \beta_1 \frac{\varpi T^{\alpha-1}}{\Gamma(\alpha)} + \beta_2 \, \lambda \Big) \leq M, \tag{3.4}$$

*then* $T(B_M(K,L)) \subseteq B_M(K,L)$.

*Proof.* Let $\psi \in B_M(K,L)$. We have

$$|(T\psi)(y)| \leq \frac{1}{\Gamma(\alpha)} \int_0^y (y-t)^{\alpha-1} \big| g(t, \psi^{[1]}(t), \psi^{[2]}(t), \ldots, \psi^{[n]}(t)) \big| \, dt + \beta_1 + \beta_2 y,$$

with

$$\begin{aligned} &\big| g\big(t, \psi^{[1]}(t), \psi^{[2]}(t), \ldots, \psi^{[n]}(t)\big) \big| \\ &\qquad = \Big| g(t, \psi^{[1]}(t), \psi^{[2]}(t), \ldots, \psi^{[n]}(t)\big) - g(t, 0, 0, \ldots, 0) + g(t, 0, 0, \ldots, 0) \Big| \\ &\quad \leq \Big| g\big(t, \psi^{[1]}(t), \psi^{[2]}(t), \ldots, \psi^{[n]}(t)\big) - g(t, 0, 0, \ldots, 0) \Big| + \sup_{t \in K} |g(t, 0, 0, \ldots, 0)| \\ &\qquad \leq \sum_{i=1}^{n} \kappa_i \|\psi^{[i]}\| + \gamma \leq \sum_{i=1}^{n} \kappa_i \sum_{j=0}^{i-1} M^j \|\psi\| + \gamma \leq L \sum_{i=1}^{n} \kappa_i \sum_{j=0}^{i-1} M^j + \gamma = \rho. \end{aligned}$$

Thus

$$|(T\psi)(y)| \leq \frac{\rho}{\Gamma(\alpha)} \int_0^y (y-t)^{\alpha-1} \, dt + \beta_1 + \beta_2 y \leq \frac{\rho}{\Gamma(\alpha+1)} T^\alpha + \beta_1 + \beta_2 T = L.$$

We then obtain $|(T\psi)(y)| \leq L$. Alternatively, since $(T\psi)(y) \geq 0$, $\forall\, y \in K$, we find $0 \leq (T\psi)(y) \leq |(T\psi)(y)| \leq L$, which proves that $T\psi \in B(K,L)$. It remains to verify that

$$\big|(T\psi)(y_2) - (T\psi)(y_1)\big| \leq M|y_2 - y_1|.$$

For $y_2 > y_1$ and $\psi \in B_M(K,L)$, we have

$$\begin{aligned}
&\big|(T\psi)(y_2) - (T\psi)(y_1)\big| \\
&\quad \le \Big| \int_0^{y_2} (y_2 - t)^{\alpha-1} E_{\alpha,\alpha}\big(-\varpi (y_2 - t)^\alpha\big) g\big(t, \psi^{[1]}(t), \psi^{[2]}(t), \dots, \psi^{[n]}(t)\big)\, dt \\
&\qquad - \int_0^{y_2} (y_1 - t)^{\alpha-1} E_{\alpha,\alpha}\big(-\varpi (y_1 - t)^\alpha\big) g\big(t, \psi^{[1]}(t), \psi^{[2]}(t), \dots, \psi^{[n]}(t)\big)\, dt \Big| \\
&\qquad + \beta_1 \big|E_\alpha(-\varpi y_2^\alpha) - E_\alpha(-\varpi y_1^\alpha)\big| + \beta_2 \big|y_2 E_{\alpha,2}(-\varpi y_2^\alpha) - y_1 E_{\alpha,2}(-\varpi y_1^\alpha)\big|.
\end{aligned}$$

Let us consider

$$\begin{aligned}
&|(T_1\psi)(y_2) - (T_1\psi)(y_1)| \\
&\quad = \Big| \int_0^{y_2} (y_2 - t)^{\alpha-1} E_{\alpha,\alpha}\big(-\varpi (y_2 - t)^\alpha\big) g\big(t, \psi^{[1]}(t), \psi^{[2]}(t), \dots, \psi^{[n]}(t)\big)\, dt \\
&\qquad - \int_0^{y_1} (y_1 - t)^{\alpha-1} E_{\alpha,\alpha}\big(-\varpi (y_1 - t)^\alpha\big) g\big(t, \psi^{[1]}(t), \psi^{[2]}(t), \dots, \psi^{[n]}(t)\big)\, dt \Big| \\
&\quad \le \int_0^{y_1} \Big| (y_2 - t)^{\alpha-1} E_{\alpha,\alpha}\big(-\varpi (y_2 - t)^\alpha\big) - (y_1 - t)^{\alpha-1} E_{\alpha,\alpha}\big(-\varpi (y_1 - t)^\alpha\big) \Big| \\
&\qquad\qquad \times \big| g\big(t, \psi^{[1]}(t), \psi^{[2]}(t), \dots, \psi^{[n]}(t)\big) \big|\, dt \\
&\qquad + \int_{y_1}^{y_2} (y_2 - t)^{\alpha-1} E_{\alpha,\alpha}\big(-\varpi (y_2 - t)^\alpha\big) \big| g\big(t, \psi^{[1]}(t), \psi^{[2]}(t), \dots, \psi^{[n]}(t)\big) \big|\, dt.
\end{aligned}$$

Using (2.5), and since $(y-t)^{\alpha-1}$ is increasing for every $\alpha \in (1,2)$, we arrive at

$$\begin{aligned}
&\big|(T_1\psi)(y_2) - (T_1\psi)(y_1)\big| \\
&\quad \le \frac{\rho}{\Gamma(\alpha)} \int_0^{y_1} \big[(y_2 - t)^{\alpha-1} - (y_1 - t)^{\alpha-1}\big]\, dt + \frac{\rho}{\Gamma(\alpha)} \int_{y_1}^{y_2} (y_2 - t)^{\alpha-1}\, dt \le \frac{\rho}{\Gamma(\alpha+1)} [y_2^\alpha - y_1^\alpha].
\end{aligned}$$

Applying the mean value theorem, we obtain

$$\big|(T_1\psi)(y_2) - (T_1\psi)(y_1)\big| \le \frac{\rho}{\Gamma(\alpha+1)} \alpha c^{\alpha-1} (y_2 - y_1), \quad c \in (y_1, y_2).$$

Hence,

$$\big|(T_1\psi)(y_2) - (T_1\psi)(y_1)\big| \le \frac{\rho}{\Gamma(\alpha)} T^{\alpha-1} |y_2 - y_1|. \tag{3.5}$$

On the other hand, let

$$\big|(T_2\psi)(y_2) - (T_2\psi)(y_1)\big| = \beta_1 \big|E_\alpha(-\varpi y_2^\alpha) - E_\alpha(-\varpi y_1^\alpha)\big| + \beta_2 \big|y_2 E_{\alpha,2}(-\varpi y_2^\alpha) - y_1 E_{\alpha,2}(-\varpi y_1^\alpha)\big|.$$

Applying the mean value theorem to $f_1(y) = E_\alpha(-\varpi y^\alpha)$, and using both (2.6) and (2.7), we deduce the following:

$$\frac{d}{dy} f_1(y) = (-\alpha\varpi y^{\alpha-1}) E^2_{\alpha,1+\alpha}(-\varpi y^\alpha) = (-\varpi y^{\alpha-1}) E_{\alpha,\alpha}(-\varpi y^\alpha).$$

Therefore, we arrive at

$$|f_1(y_2) - f_1(y_1)| \le \frac{\varpi T^{\alpha-1}}{\Gamma(\alpha)} |y_2 - y_1|.$$

Applying the same procedure to $f_2(y) = yE_{\alpha,2}(-\varpi y^\alpha)$ with (2.4), we find

$$\frac{d}{dy} f_2(y) = E_{\alpha,2}(-\varpi y^\alpha) - \alpha\varpi y^\alpha E^2_{\alpha,2+\alpha}(-\varpi y^\alpha)$$
$$= E_{\alpha,2}(-\varpi y^\alpha) - \varpi y^\alpha \big(E_{\alpha,2+\alpha}(-\varpi y^\alpha) + E_{\alpha,1+\alpha}(-\varpi y^\alpha)\big).$$

It follows that

$$|f_2(y_2) - f_2(y_1)| \le \Big[1 + \varpi T^\alpha \Big(\frac{1}{\Gamma(\alpha+2)} + \frac{1}{\Gamma(\alpha+1)}\Big)\Big] |y_2 - y_1|.$$

Accordingly, we derive

$$\big|(T_2\psi)(y_2) - (T_2\psi)(y_1)\big| \le \Big(\beta_1 \frac{\varpi T^{\alpha-1}}{\Gamma(\alpha)} + \beta_2\, \lambda\Big) |y_2 - y_1|. \tag{3.6}$$

Combining both results (**??**) and (**??**), we arrive at

$$\big|(T\psi)(y_2) - (T\psi)(y_1)\big| \le \Big(\frac{\rho}{\Gamma(\alpha)} T^{\alpha-1} + \beta_1 \frac{\varpi T^{\alpha-1}}{\Gamma(\alpha)} + \beta_2\lambda\Big)|y_2 - y_1| \le M|y_2 - y_1|,$$

which confirms that $T\psi \in B_M(K, L)$. □

**Theorem 3.1.** *Assume that conditions (2.1), (3.1) and (3.2) hold. Then there exists at least one mild nonnegative solution to problem (1.1).*

*Proof.* According to Lemma 2.1, problem (1.1) admits a mild solution in $B_M(K, L)$ if and only if the operator $T$, introduced in (**??**), has a fixed point. The Schauder fixed point theorem is fulfilled from Lemma 3.1 and Lemma 3.2. As a result, $T$ has at least one fixed point in $B_M(K, L)$, which represents a mild nonnegative solution to problem (1.1). □

## 3.2 Uniqueness

**Theorem 3.2.** *Building on the assumptions of Theorem 3.1, suppose that*

$$\frac{T^\alpha}{\Gamma(\alpha+1)} \sum_{i=1}^{n} \kappa_i \sum_{j=0}^{i-1} M^j < 1. \tag{3.7}$$

*Then the operator $T$ admits a unique fixed point which corresponds to the unique mild nonnegative solution of problem (1.1) in $B_M(K, L)$.*

*Proof.* Suppose that problem (1.1) admits two distinct mild solutions $\psi$ and $\varphi$. Using inequalities (**??**), (**??**), it can be inferred that

$$|\psi(y) - \varphi(y)| = \big|(T\psi)(y) - (T\varphi)(y)\big| \le \frac{T^\alpha}{\Gamma(\alpha+1)} \sum_{i=1}^{n} \kappa_i \sum_{j=0}^{i-1} M^j \|\psi - \varphi\| < \|\psi - \varphi\|.$$

Hence, $\|\psi - \varphi\| < \|\psi - \varphi\|$. Thus, a contradiction arises, leading to the conclusion that $T$ has a unique fixed point, which represents the unique mild nonnegative solution of (1.1). □

**Example 3.1.** Consider the following semi-linear iterative fractional relaxation initial value problem:

$$\begin{cases} {}^C D^{\frac{3}{2}}_{0^+}\psi(y) + \psi(y) = \frac{1}{12}\, y^2 + \frac{1}{15}\, \sin^2(y)\psi^{[1]}(y) + \frac{1}{10}\, \cos^2(y)\psi^{[2]}(y), \quad y \in \Big[0, \frac{1}{2}\Big], \\ \psi(0) = \frac{1}{5}\,, \quad \psi'(0) = \frac{1}{2}\,, \end{cases} \tag{3.8}$$

where $\alpha = \frac{3}{2}$, $T = \frac{1}{2}$ and

$$g(y, \psi_1, \psi_2) = \frac{1}{12}\, y + \frac{1}{15}\, \sin^2(y)\psi_1 + \frac{1}{10}\, \cos^2(y)\psi_2.$$

Let

$$B_M(K, L) = \Big\{\psi \in B(K, L) : \ |\psi(y_2) - \psi(y_1)| \le M|y_2 - y_1|\Big\},$$

where

$$B(K, L) = \big\{\psi \in C(K, \mathbb{R}) : \ 0 \le \psi(y) \le L \le T\big\}.$$

For $L = 0.5$ and $M = 1.5$, we have

$$\begin{aligned}\big|g(y, \psi_1, \psi_2) - g(y, \phi_1, \phi_2)\big| &= \Big|\frac{1}{15}\, \sin(y)\psi_1 + \frac{1}{10}\, \cos^2(y)\psi_2 - \frac{1}{15}\, \sin(y)\phi_1 - \frac{1}{10}\, \cos^2(y)\phi_2\Big| \\ &= \Big|\frac{1}{15}\, \sin(y)(\psi_1 - \phi_1) + \frac{1}{10}\, \cos^2(y)(\psi_2 - \phi_2)\Big| \le \frac{1}{15}\, |\psi_1 - \phi_1| + \frac{1}{10}\, |\psi_2 - \phi_2|.\end{aligned}$$

Then

$$\big|g(y, \psi_1, \psi_2) - g(y, \phi_1, \phi_2)\big| \le \sum_{i=1}^{2} \kappa_i |\psi_i - \phi_i|,$$

with $\kappa_1 = \frac{1}{15}$ and $\kappa_2 = \frac{1}{10}$, then $g$ satisfies condition (2.1).

Additionally, we have

$$\begin{aligned}\frac{\rho}{\Gamma(\alpha+1)}\, T^{\alpha} + \beta_1 + \beta_2 T &= \frac{\sup_{y\in K} |g(y, 0, 0)| + L(\kappa_1 + \kappa_2(1 + M))}{\Gamma(\alpha+1)}\, T^{\frac{3}{2}} + \beta_1 + \beta_2 T \\ &= \frac{\frac{1}{48} + \frac{1}{2}\,(\frac{1}{15} + \frac{2.5}{10})}{\Gamma(\frac{5}{2})} \Big(\frac{1}{2}\Big)^{\frac{3}{2}} + \frac{1}{5} + \frac{1}{4} \simeq 0.497 \le L = 0.5\end{aligned}$$

and

$$\begin{aligned}&\frac{\rho}{\Gamma(\alpha)}\, T^{\alpha-1} + \beta_1\, \frac{\varpi T^{\alpha-1}}{\Gamma(\alpha)} + \beta_2 \lambda \\ &= \frac{\sup_{y\in K} |g(y, 0, 0)| + L \sum_{i=1}^{n} \kappa_i \sum_{j=0}^{i-1} M^j}{\Gamma(\frac{3}{2})}\, T^{\alpha-1} + \beta_1\, \frac{\varpi T^{\alpha-1}}{\Gamma(\frac{3}{2})} + \beta_2 \Big(1 + \varpi T^{\alpha}\Big(\frac{1}{\Gamma(\frac{7}{2})} + \frac{1}{\Gamma(\frac{5}{2})}\Big)\Big) \\ &= \frac{\frac{1}{48} + \frac{1}{2}\,(\frac{1}{15} + \frac{2.5}{10}) + \frac{1}{5}}{\Gamma(\frac{3}{2})} \sqrt{\frac{1}{2}} + \frac{1}{2}\Big(1 + \sqrt{\frac{1}{2}}\,\Big(\frac{1}{\Gamma(\frac{7}{2})} + \frac{1}{\Gamma(\frac{5}{2})}\Big)\Big) \simeq 1.174 \le M = 1.5.\end{aligned}$$

Also, we have

$$\frac{T^{\alpha}}{\Gamma(\alpha+1)} \sum_{i=1}^{n} \kappa_i \sum_{j=0}^{i-1} M^j = \frac{(\frac{1}{2})^{\frac{3}{2}}}{\Gamma(\frac{5}{2})} \Big(\frac{1}{15} + \frac{2.5}{10}\Big) \simeq 0.0842 < 1.$$

Then, according to Theorem 3.1 and Theorem 3.2, problem (3.3) admits a unique mild nonnegative solution in $B_M(K, L)$.

## 3.3 Continuous dependence

**Definition 3.1** ([11]). A mild nonnegative solution of problem (1.1) is said to depend continuously on the initial data and the function $g$ if small changes in these components lead to small changes in the solution.

**Theorem 3.3.** *Assume the conditions of Theorem 3.2 are satisfied. The unique mild nonnegative solution of problem (1.1) depends continuously on $g$, $\beta_1$ and $\beta_2$.*

*Proof.* Let $g,\widetilde{g}\in C(K\times\mathbb{R}^n,\mathbb{R}_+)$ and $\beta_1,\ \widetilde{\beta}_1,\ \beta_2,\ \widetilde{\beta}_2$ be positive real numbers. We consider two solutions $\psi$ and $\widetilde{\psi}$ such that

$$\psi(y)=\int_0^y (y-t)^{\alpha-1}E_{\alpha,\alpha}\big(-\varpi(y-t)^{\alpha}\big)g\big(t,\psi^{[1]}(t),\psi^{[2]}(t),\dots,\psi^{[n]}(t)\big)\,dt + \beta_1E_{\alpha}(-\varpi y^{\alpha})+\beta_2 yE_{\alpha,2}(-\varpi y^{\alpha})$$

and

$$\widetilde{\psi}(y)=\int_0^y (y-t)^{\alpha-1}E_{\alpha,\alpha}\big(-\varpi(y-t)^{\alpha}\big)g\big(t,\widetilde{\psi}^{[1]}(t),\widetilde{\psi}^{[2]}(t),\dots,\widetilde{\psi}^{[n]}(t)\big)\,dt + \widetilde{\beta}_1E_{\alpha}(-\varpi y^{\alpha})+\widetilde{\beta}_2 yE_{\alpha,2}(-\varpi y^{\alpha}).$$

We have

$$\begin{aligned}|\psi(y)-\widetilde{\psi}(y)| &\le \frac{1}{\Gamma(\alpha)}\int_0^y (y-t)^{\alpha-1}\Big|g\big(t,\psi^{[1]}(t),\psi^{[2]}(t),\dots,\psi^{[n]}(t)\big)-g\big(t,\widetilde{\psi}^{[1]}(t),\widetilde{\psi}^{[2]}(t),\dots,\widetilde{\psi}^{[n]}(t)\big)\Big|\,dt\\ &\quad+|\beta_1-\widetilde{\beta}_1|+|\beta_2-\widetilde{\beta}_2|T\end{aligned}$$

with

$$\begin{aligned}&\Big|g\big(t,\psi^{[1]}(t),\psi^{[2]}(t),\dots,\psi^{[n]}(t)\big)-\widetilde{g}\big(t,\widetilde{\psi}^{[1]}(t),\widetilde{\psi}^{[2]}(t),\dots,\widetilde{\psi}^{[n]}(t)\big)\Big|\\ &\quad=\Big|g\big(t,\psi^{[1]}(t),\psi^{[2]}(t),\dots,\psi^{[n]}(t)\big)-\widetilde{g}\big(t,\psi^{[1]}(t),\psi^{[2]}(t),\dots,\psi^{[n]}(t)\big)\\ &\qquad+\widetilde{g}\big(t,\psi^{[1]}(t),\psi^{[2]}(t),\dots,\psi^{[n]}(t)\big)-\widetilde{g}\big(t,\widetilde{\psi}^{[1]}(t),\widetilde{\psi}^{[2]}(t),\dots,\widetilde{\psi}^{[n]}(t)\big)\Big|.\end{aligned}$$

Using (2.1) and Lemma 2.2, we obtain

$$\Big|g\big(t,\psi^{[1]}(t),\psi^{[2]}(t),\dots,\psi^{[n]}(t)\big)-\widetilde{g}\big(t,\widetilde{\psi}^{[1]}(t),\widetilde{\psi}^{[2]}(t),\dots,\widetilde{\psi}^{[n]}(t)\big)\Big| \le \|g-\widetilde{g}\|+\sum_{i=1}^{n}\kappa_i\sum_{j=0}^{i-1}M^j\|\psi-\widetilde{\psi}\|.$$

Then

$$\begin{aligned}|\psi(y)-\widetilde{\psi}(y)|&\le\frac{1}{\Gamma(\alpha)}\int_0^y(y-t)^{\alpha-1}\Big(\|g-\widetilde{g}\|+\sum_{i=1}^{n}\kappa_i\sum_{j=0}^{i-1}M^j\|\psi-\widetilde{\psi}\|\Big)+|\beta_1-\widetilde{\beta}_1|+|\beta_2-\widetilde{\beta}_2|T\\ &\le\frac{T^{\alpha}}{\Gamma(\alpha+1)}\|g-\widetilde{g}\|+\frac{T^{\alpha}}{\Gamma(\alpha+1)}\sum_{i=1}^{n}\kappa_i\sum_{j=0}^{i-1}M^j\|\psi-\widetilde{\psi}\|+|\beta_1-\widetilde{\beta}_1|+|\beta_2-\widetilde{\beta}_2|T.\end{aligned}$$

Thus we arrive at

$$\|\psi-\widetilde{\psi}\|\le\frac{T^{\alpha}}{\Gamma(\alpha+1)}\|g-\widetilde{g}\|+\frac{T^{\alpha}}{\Gamma(\alpha+1)}\sum_{i=1}^{n}\kappa_i\sum_{j=0}^{i-1}M^j\|\psi-\widetilde{\psi}\|+|\beta_1-\widetilde{\beta}_1|+T|\beta_2-\widetilde{\beta}_2|.$$

Hence,

$$\|\psi-\widetilde{\psi}\| \leq \frac{\frac{T^{\alpha}}{\Gamma(\alpha+1)}}{1-\frac{T^{\alpha}}{\Gamma(\alpha+1)}\sum\limits_{i=1}^{n}\kappa_i\sum\limits_{j=0}^{i-1}M^j}\|g-\widetilde{g}\|$$
$$+\frac{1}{1-\frac{T^{\alpha}}{\Gamma(\alpha+1)}\sum\limits_{i=1}^{n}\kappa_i\sum\limits_{j=0}^{i-1}M^j}|\beta_1-\widetilde{\beta}_1|+\frac{T}{1-\frac{T^{\alpha}}{\Gamma(\alpha+1)}\sum\limits_{i=1}^{n}\kappa_i\sum\limits_{j=0}^{i-1}M^j}|\beta_2-\widetilde{\beta}_2|. \qquad \square$$

## 3.4 Stability

**Definition 3.2** ([17]). Problem (1.1) is considered to be Ulam–Hyers stable if, for every $\varepsilon>0$ and for every $\widetilde{\psi}\in B_M(K,L)$ satisfying

$$\left|{}^{C}D_{0^+}^{\alpha}\widetilde{\psi}(y)+\varpi\widetilde{\psi}(y)-g\big(y,\widetilde{\psi}^{[1]}(y),\widetilde{\psi}^{[2]}(y),\dots,\widetilde{\psi}^{[n]}(y)\big)\right|\leq\varepsilon,\quad y\in K,$$

with $\widetilde{\psi}(0)=\beta_1$, $\widetilde{\psi}'(0)=\beta_2$, there exists a mild nonnegative solution $\psi\in B_M(K,L)$ of problem (1.1) such that $|\widetilde{\psi}(y)-\psi(y)|\leq\varepsilon C_g$, $y\in K$, for some constant $C_g>0$.

**Theorem 3.4.** *Suppose that the assumptions of Theorem 3.2 are satisfied. Then problem (1.1) is stable in the Ulam–Hyres sense.*

*Proof.* Let $\psi\in B_M(K,L)$ be the unique mild nonnegative solution of problem (1.1). Then, it follows from Lemma 2.1 that for $y\in K$,

$$\psi(y)=\int_0^y (y-t)^{\alpha-1}E_{\alpha,\alpha}(-\varpi(y-t)^{\alpha})g\big(t,\psi^{[1]}(t),\psi^{[2]}(t),\dots,\psi^{[n]}(t)\big)\,dt$$
$$+\beta_1E_{\alpha}(-\varpi y^{\alpha})+\beta_2yE_{\alpha,2}(-\varpi y^{\alpha}).$$

And let $\widetilde{\psi}(y)$ be the approximate solution of (1.1) satisfying

$$\Bigg|\widetilde{\psi}(y)-\int_0^y (y-t)^{\alpha-1}E_{\alpha,\alpha}\big(-\varpi(y-t)^{\alpha}\big)g\big(t,\widetilde{\psi}^{[1]}(t),\widetilde{\psi}^{[2]}(t),\dots,\widetilde{\psi}^{[n]}(t)\big)\,dt$$
$$-\beta_1E_{\alpha}(-\varpi y^{\alpha})-\beta_2yE_{\alpha,2}(-\varpi y^{\alpha})\Bigg|\leq\varepsilon.$$

For every $y\in K$, we have

$$\begin{aligned}|\widetilde{\psi}(y)-\psi(y)| &= \Bigg|\widetilde{\psi}(y)-\int_0^y (y-t)^{\alpha-1}E_{\alpha,\alpha}\big(-\varpi(y-t)^{\alpha}\big)g\big(t,\psi^{[1]}(t),\psi^{[2]}(t),\dots,\psi^{[n]}(t)\big)\,dt\\ &\qquad -\beta_1E_{\alpha}(-\varpi y^{\alpha})-\beta_2yE_{\alpha,2}(-\varpi y^{\alpha})\Bigg|\\ &\leq \Bigg|\widetilde{\psi}(y)-\int_0^y (y-t)^{\alpha-1}E_{\alpha,\alpha}\big(-\varpi(y-t)^{\alpha}\big)g\big(t,\widetilde{\psi}^{[1]}(t),\widetilde{\psi}^{[2]}(t),\dots,\widetilde{\psi}^{[n]}(t)\big)\,dt\\ &\qquad -\beta_1E_{\alpha}(-\varpi y^{\alpha})-\beta_2yE_{\alpha,2}(-\varpi y^{\alpha})\Bigg|\\ &\quad +\int_0^y (y-t)^{\alpha-1}E_{\alpha,\alpha}\big(-\varpi(y-t)^{\alpha}\big)\Big|g\big(s,\widetilde{\psi}^{[1]}(s),\widetilde{\psi}^{[2]}(s),\dots,\widetilde{\psi}^{[n]}(s)\big)\end{aligned}$$

$$
\begin{aligned}
&\qquad\qquad\qquad\qquad - g\big(s, \psi^{[1]}(s), \psi^{[2]}(s), \dots, \psi^{[n]}(s)\big)\Big| \, ds \\
&\le \varepsilon + \frac{1}{\Gamma(\alpha)} \int_0^y (y-s)^{\alpha-1} \Big| g\big(t, \widetilde{\psi}^{[1]}(t), \widetilde{\psi}^{[2]}(t), \dots, \widetilde{\psi}^{[n]}(t)\big) \, dt \\
&\qquad\qquad\qquad\qquad - g\big(t, \psi^{[1]}(t), \psi^{[2]}(t), \dots, \psi^{[n]}(t)\big)\Big| \, dt \\
&\le \epsilon + \frac{T^\alpha}{\Gamma(\alpha+1)} \sum_{i=1}^{n} \kappa_i \sum_{j=0}^{i-1} M^j \|\widetilde{\psi} - \psi\|.
\end{aligned}
$$

Then

$$
\|\widetilde{\psi} - \psi\| \le \epsilon + \frac{T^\alpha}{\Gamma(\alpha+1)} \sum_{i=1}^{n} \kappa_i \sum_{j=0}^{i-1} M^j \|\widetilde{\psi} - \psi\|.
$$

As a consequence, we obtain

$$
|\widetilde{\psi}(y) - \psi(y)| \le \|\widetilde{\psi} - \psi\| \le \varepsilon \left[ \frac{1}{1 - \frac{T^\alpha}{\Gamma(\alpha+1)} \sum\limits_{i=1}^{n} \kappa_i \sum\limits_{j=0}^{i-1} M^j} \right],
$$

which leads to $|\widetilde{\psi}(y) - \psi(y)| \le \varepsilon C_g$, $y \in K$, thereby proving the theorem. □

**Example 3.2.** We consider the problem

$$
\begin{cases}
{}^{C}D_{0^+}^{\frac{3}{2}} \psi(y) + \psi(y) \\
\qquad = \dfrac{1}{9} + \dfrac{1}{7} y^3 + \dfrac{1}{18} \sin(y) \psi^{[1]}(y) + \dfrac{1}{19} \cos(y) \psi^{[2]}(y) + \dfrac{1}{4} y^2 \psi^{[3]}(y), \quad y \in \Big[0, \dfrac{1}{2}\Big], \\
\psi(0) = \dfrac{1}{5}, \quad \psi'(0) = \dfrac{1}{3},
\end{cases} \tag{3.9}
$$

where $\alpha = \frac{3}{2}$ $T = \frac{1}{2}$, $\varpi = 1$ and

$$
g(y, \psi_1, \psi_2, \psi_3) = \frac{1}{9} + \frac{1}{7} y^3 + \frac{1}{18} \sin(y) \psi_1(y) + \frac{1}{19} \cos(y) \psi_2(y) + \frac{1}{7} y^2 \psi_3(y).
$$

Let

$$
B_M(K, L) = \Big\{ \psi \in B(K, L) : \ |\psi(x_2) - \psi(x_1)| \le M |x_2 - x_1| \Big\},
$$

where

$$
B(K, L) = \big\{ \psi \in C(K, \mathbb{R}) : \ 0 \le \psi(x) \le L \le T \big\}.
$$

For $L = 0.5$ and $M = 1$, we have

$$
\begin{aligned}
&|g(y, \psi_1, \psi_2, \psi_3) - g(y, \phi_1, \phi_2, \phi_3)| \\
&\qquad = \Big| \frac{1}{18} \sin(y)(\psi_1 - \phi_1) + \frac{1}{19} \cos(y)(\psi_2 - \phi_2) + \frac{1}{7} y^2 (\psi_3 - \phi_3) \Big| \\
&\qquad\qquad \le \frac{1}{18} |\psi_1 - \phi_1| + \frac{1}{19} |\psi_2 - \phi_2| + \frac{1}{14} |\psi_3 - \phi_3|.
\end{aligned}
$$

Then

$$
\big| g(y, \psi_1, \psi_2, \psi_3) - g(y, \phi_1, \phi_2, \phi_3) \,| \le \sum_{i=1}^{3} \kappa_i |\psi_i - \phi_i|,
$$

with $\kappa_1 = \frac{1}{18}$, $\kappa_2 = \frac{1}{19}$ and $\kappa_3 = \frac{1}{14}$, and $g$ satisfies condition (2.1).

Additionally, we have

$$\begin{aligned}\frac{\rho}{\Gamma(\alpha+1)}T^{\alpha}+\beta_1+\beta_2T &= \frac{\gamma+L\sum\limits_{i=1}^{n}\kappa_i\sum\limits_{j=0}^{i-1}M^j}{\Gamma(\alpha+1)}T^{\frac{3}{2}}+\beta_1+\beta_2T\\ &= \frac{\sup\limits_{y\in K}|g(y,0,0)|+L(\kappa_1+\kappa_2(1+M)+\kappa_3(1+M+M^2))}{\Gamma(\frac{5}{2})}T^{\frac{3}{2}}+\beta_1+\beta_2T\\ &= \frac{\frac{65}{504}+\frac{1}{2}\left(\frac{1}{18}+\frac{2}{19}+\frac{3}{14}\right)}{\Gamma(\frac{5}{2})}\left(\frac{1}{2}\right)^{\frac{3}{2}}+\frac{1}{5}+\frac{1}{6}\simeq 0.450\le L=0.5\end{aligned}$$

and

$$\begin{aligned}&\frac{\rho}{\Gamma(\alpha)}T^{\alpha-1}+\beta_1\frac{\varpi T^{\alpha-1}}{\Gamma(\alpha)}+\beta_2\lambda\\ &= \frac{\rho}{\Gamma(\alpha)}T^{\alpha-1}+\beta_1\frac{\varpi T^{\alpha-1}}{\Gamma(\alpha)}+\beta_2\Big(1+\varpi T^{\alpha}\Big(\frac{1}{\Gamma(\alpha+2)}+\frac{1}{\Gamma(\alpha+1)}\Big)\Big)\\ &= \frac{\sup\limits_{y\in K}|g(y,0,0)|+L\sum\limits_{i=1}^{n}\kappa_i\sum\limits_{j=0}^{i-1}M^j}{\Gamma(\alpha)}T^{\alpha-1}+\beta_1\frac{\varpi T^{\alpha-1}}{\Gamma(\alpha)}+\beta_2\Big(1+\varpi T^{\alpha}\Big(\frac{1}{\Gamma(\alpha+2)}+\frac{1}{\Gamma(\alpha+1)}\Big)\Big)\\ &= \frac{\frac{65}{504}+\frac{1}{2}\left(\frac{1}{18}+\frac{2}{19}+\frac{3}{14}\right)+\frac{1}{5}}{\Gamma(\frac{3}{2})}\sqrt{\frac{1}{2}}+\frac{1}{3}\Big(1+\sqrt{\frac{1}{2}}\Big(\frac{1}{\Gamma(\frac{7}{2})}+\frac{1}{\Gamma(\frac{5}{2})}\Big)\Big)\simeq 0.993\le 1=M.\end{aligned}$$

Also, we have

$$\begin{aligned}\frac{T^{\alpha}}{\Gamma(\alpha+1)}\sum_{i=1}^{n}\kappa_i\sum_{j=0}^{i-1}M^j &= \frac{(\frac{1}{2})^{\frac{3}{2}}}{\Gamma(\frac{5}{2})}\sum_{i=1}^{3}\kappa_i\sum_{j=0}^{i-1}M^j\\ &= \frac{(\frac{1}{2})^{\frac{3}{2}}}{\Gamma(\frac{5}{2})}\left(\kappa_1+\kappa_2(1+M)+\kappa_3(1+M+M^2)\right)=\frac{(\frac{1}{2})^{\frac{3}{2}}}{\Gamma(\frac{5}{2})}\left(\frac{1}{18}+\frac{2}{19}+\frac{3}{14}\right)\simeq 0.099<1.\end{aligned}$$

Consequently, from both Theorem 3.1 and Theorem 3.2, we deduce that problem (3.4) admits a unique mild nonnegative solution. Additionally, this solution depends continuously on the components $g$, $\beta_1$ and $\beta_2$. Furthermore, Theorem 3.4 confirms the stability of the solution in the Ulam–Hyres sense.

# 4 Conclusion

This study has addressed a semi-linear Caputo iterative fractional relaxation differential equation with non-zero initial conditions. We investigated the existence, uniqueness and continuous dependence of the mild nonnegative solution along with its Ulam–Hyers stability. These findings contribute to the theoretical development of such equations and lay the groundwork for future research.

As a future research, the qualitative properties of solutions for nonlinear $\psi$-Caputo iterative fractional relaxation differential equations might be considered.

# References

[1] S. Abbas, B. Ahmad, M. Benchohra and A. Salim, *Fractional Difference, Differential Equations, and Inclusions—Analysis and Stability*. Morgan Kaufmann Publishers, Palo Alto, CA, 2024.

[2] R. P. Agarwal, M. Meehan and D. O'Regan, *Fixed Point Theory and Applications*. Cambridge Tracts in Mathematics, 141. Cambridge University Press, Cambridge, 2001.

[3] Ph. N. A. Akuka, B. Seidu, E. Okyere and S. Abagna, Fractional-order epidemic model for measles infection. *Scientifica* **2024**, Article ID 8997302, 18 pp.

[4] K. Balachandran, *An Introduction to Fractional Differential Equations.* Industrial and Applied Mathematics. Springer, Singapore, [2023

[5] M. A. Balcı, Fractional virus epidemic model on financial networks. *Open Math.* **14** (2016), no. 1, 1074–1086.

[6] M. Benchohra, S. Bouriah, A. Salim and Y. Zhou, *Fractional Differential Equations – a Coincidence Degree Approach.* Fractional Calculus in Applied Sciences and Engineering, 12. De Gruyter, Berlin, [2024],

[7] M. Benchohra, E. Karapınar, J. E. Lazreg and A. Salim, *Advanced Topics in Fractional Differential Equations – a Fixed Point Approach.* Synthesis Lectures on Mathematics and Statistics. Springer, Cham, [2023], ©2023

[8] M. Benchohra, E. Karapınar, J. E. Lazreg and A. Salim, *Fractional Differential Equations – New Advancements for Generalized Fractional Derivatives.* Synthesis Lectures on Mathematics and Statistics. Springer, Cham, 2023.

[9] Sh. Das, *Functional Fractional Calculus.* Second edition. Springer-Verlag, Berlin, 2011.

[10] L. Debnath and D. Bhatta, *Integral Transforms and their Applications.* 2nd Edition. Chapman and Hall/CRC, New York, 2016.

[11] K. Diethelm,An extension of the well-posedness concept for fractional differential equations of Caputo's type. *Appl. Anal.* **93** (2014), no. 10, 2126–2135.

[12] A. Chidouh, A. Guezane-Lakoud and R. Bebbouchi, Positive solutions of the fractional relaxation equation using lower and upper solutions. *Vietnam J. Math.* **44** (2016), no. 4, 739–748.

[13] R. Gorenflo, A. A. Kilbas, F. Mainardi and S. Rogosin, *Mittag–Leffler Functions, Related Topics and Applications.* Second edition [of 3244285]. Springer Monographs in Mathematics. Springer, Berlin, 2020.

[14] A. Guerfi and A. Ardjouni, Mild nonnegative solutions for fractional iterative differential equations. *Asia Pac. J. Math.* **10** (2023), no. 6, 9 pp.

[15] A. Guerfi and A. Ardjouni, Existence, uniqueness, continuous dependence and Ulam stability of mild solutions for an iterative fractional differential equation. *Cubo* **24** (2022), no. 1, 83–94.

[16] A. A. Kilbas, H. M. Srivastava and J. J. Trujillo, *Theory and Applications of Fractional Differential Equations.* North-Holland Mathematics Studies, 204. Elsevier Science B.V., Amsterdam, 2006.

[17] H. Liu and Y. Li, Hyers–Ulam stability of linear fractional differential equations with variable coefficients. *Adv. Difference Equ.* **2020**, Paper no. 404, 10 pp.

[18] K. S. Miller and B. Ross, *An Introduction to the Fractional Calculus and Fractional Differential Equations.* A Wiley-Interscience Publication. John Wiley & Sons, Inc., New York, 1993.

[19] I. Podlubny, *Fractional Differential Equations. An Introduction to Fractional Derivatives, Fractional Differential Equations, to Methods of their Solution and some of their Applications.* Mathematics in Science and Engineering, 198. Academic Press, Inc., San Diego, CA, 1999.

[20] W. E. Raslan, Fractional mathematical modeling for epidemic prediction of COVID-19 in Egypt. *Ain Shams Engineering Journal* **12** (2021), no. 3, 3057–3062.

[21] S. S. Ray and A. K. Gupta, *Wavelet Methods for Solving Partial Differential Equations and Fractional Differential Equations.* CRC Press, Boca Raton, FL, 2018.

[22] A. Salim and M. Benchohra, Existence and uniqueness results for generalized Caputo iterative fractional boundary value problems. *Fract. Differ. Calc.* **12** (2022), no. 2, 197–208.

[23] J. L. Schiff, *The Laplace Transform. Theory and Applications.* Springer-Verlag, New York, Berlin, Heidelberg, 1999.

[24] J. Singh, J. Y. Hristov and Z. Hammouch (Eds.), *New Trends in Fractional Differential Equations with Real-World Applications in Physics.* Frontiers Media SA, 2020.

[25] J. Wang, M. Fečkan and Y. Zhou, Presentation of solutions of impulsive fractional Langevin equations and existence results. *The European Physical Journal Special Topics* **222** (2013), 1857–1874.

[26] H. Y. Zhao and J. Liu, Periodic solutions of an iterative functional differential equation with variable coefficients. *Math. Methods Appl. Sci.* **40** (2017), no. 1, 286–292.

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