

# Memoirs on Differential Equations and Mathematical Physics

VOLUME 95, 2025, 1–18

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A STUDY OF A SYSTEM OF NONLINEAR  
CAPUTO FRACTIONAL DIFFERENTIAL EQUATIONS  
WITH NEW INTEGRAL BOUNDARY CONDITIONS

**Abstract.** We investigate the existence of solutions for a coupled system of nonlinear Caputo fractional differential equations equipped with a new class of integral boundary conditions. The existence of solutions for the given problem is shown with the aid of the Leray–Schauder nonlinear alternative, while the Banach fixed point theorem is applied to establish a uniqueness result for the given problem. An example is formulated to demonstrate the application of main results.

**2020 Mathematics Subject Classification.** 34A08, 34B10, 34B15.

**Key words and phrases.** Caputo fractional derivative, system, integral boundary conditions, existence, fixed point

# 1 Introduction

In this paper, we consider a new boundary value problem consisting of a system of nonlinear Caputo fractional differential equations and closed type integral boundary conditions. Precisely, we discuss the existence of solutions for the following problem:

$$\begin{cases} {}^C D^{q_1} \varphi(t) = \rho_1(t, \varphi(t), \psi(t)), & 2 < q_1 < 3, \quad t \in \mathcal{J} = [0, T], \quad T > 0, \\ {}^C D^{q_2} \psi(t) = \rho_2(t, \varphi(t), \psi(t)), & 2 < q_2 < 3, \quad t \in \mathcal{J} = [0, T], \quad T > 0, \end{cases} \quad (1.1)$$

$$\begin{cases} \varphi(0) = 0, \quad \varphi'(T) = 0, \quad \varphi(T) = \int_0^\xi [p_1 \psi(s) + T p_2 \psi'(s)] ds, \quad p_1, p_2 \in \mathbb{R}, \\ \psi(0) = 0, \quad \psi'(T) = 0, \quad \psi(T) = \int_0^\xi [r_1 \varphi(s) + T r_2 \varphi'(s)] ds, \quad r_1, r_2 \in \mathbb{R}, \end{cases} \quad (1.2)$$

where  ${}^C D^{q_i}$  denotes the Caputo fractional derivatives of order  $q_i$  ( $i = 1, 2$ ) (here,  ${}^C D^{q_i}$  is used instead of  ${}^C D_{0+}^{q_i}$  for the sake of convenience),  $\rho_1, \rho_2 \in C(\mathcal{J} \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$  and  $\xi \in (0, T)$ .

Here, we emphasize that the first integral boundary condition in (1.2) can be interpreted as the value of the unknown function  $\varphi$  at the right end  $T$  of the interval  $[0, T]$ , is proportional to the linear combination of the unknown function  $\psi$  and its derivative  $\psi'$  on the sub-interval  $[0, \xi]$  of  $[0, T]$ . The second integral boundary condition can be interpreted in a similar manner by interchanging the role of  $\varphi$  and  $\psi$ . It is worthwhile to mention that the involvement of both unknown functions and their derivatives in the integral boundary conditions makes the proposed problem novel and useful. It can be noticed that the integral boundary conditions given in (1.2) can be written as

$$\varphi(T) = p_1 \int_0^{\xi-} \psi(s) ds + T p_2 \psi(\xi), \quad \psi(T) = r_1 \int_0^{\xi-} \varphi(s) ds + T r_2 \varphi(\xi).$$

Now, we illustrate the application of integral boundary conditions and systems of fractional differential equations. Concerning the importance of integral boundary conditions, it is imperative to point out that such boundary conditions play an important role in the investigation of problems of applied nature. In case of fluid flow problems [32, 36, 45], the idea of integral boundary conditions serves as an effective tool to describe the boundary data on arbitrary shaped structures as the assumption of circular cross-section cannot be justified on such (curved) structures. Furthermore, integral boundary conditions appear in the study of thermal conduction [8], semiconductor [22], hydrodynamic problems [10], diffusion problems [7], bacterial self-regularization [11], etc. For some interesting results on boundary value problems with integral boundary conditions, see, e.g., [3, 20, 38] and the references cited therein.

Let us now review some recent literature on fractional differential equations. In recent years, there has been observed a great surge in developing the topic of fractional calculus because of its extensive application in the mathematical modeling of several scientific and technical phenomena. Examples include immune system with memory [13], co-infection of malaria and HIV/AIDS [9], chaos and fractional dynamics [43], stabilization of chaotic systems [15], chaotic synchronization [41, 44], dynamical networks with multiple weights [42], neural networks [34], phytoplankton-zooplankton systems [24], economic model [35], financial economics [16], etc.

As the mathematical models associated with physical phenomena contain initial and boundary value problems, many researchers have shown a keen interest in developing the theoretical aspects of such problems (see, e.g., the books [4, 5] and a series of research articles [1, 23, 26, 28, 29, 33, 39]). Keeping in mind the occurrence of fractional differential systems in the mathematical modeling of several real world problems [12, 19, 37, 40], many investigators studied such systems with a variety of boundary conditions (see, e.g., [14, 18, 21, 25, 30, 31]). For some works on fractional differential systems with closed boundary conditions, see, e.g., [2, 6] and the references cited therein.

Our study in the present research paper is motivated by the application of integral boundary conditions and systems of fractional differential equations in a variety of physical and technical situations.

We prove the existence and uniqueness of solutions to the problem (1.1), (1.2) with the aid of standard fixed point theorems. The work established in this paper is new and contributes to the literature on fractional-order systems with integral boundary conditions.

The remainder of the paper is arranged as follows. We collect some preliminary definitions of fractional calculus and solve a linear version of problem (1.1), (1.2) in Section 2. The main results and examples illustrating these results are presented in Section 3. The concluding remarks and some special cases arising from the present study are elaborated in Section 4.

## 2 A preliminary result

Before proceeding for a preliminary result dealing with the linear version of problem (1.1), (1.2), we enlist the related definitions from fractional calculus.

**Definition 2.1** ([27]). For a locally integrable real-valued function  $\sigma$  defined on  $-\infty \leq a < t < b \leq +\infty$ , the (left) Riemann–Liouville fractional integral of order  $\omega \in \mathbb{R}^+$ , denoted by  $I_{a+}^\omega \sigma$ , is given by

$$I_{a+}^\omega \sigma(t) = \frac{1}{\Gamma(\omega)} \int_a^t (t-s)^{\omega-1} \sigma(s) ds,$$

where  $\Gamma$  denotes the Euler gamma function.

**Remark 2.1.** For  $\alpha, \beta \in \mathbb{R}^+$ ,  $x \in [a, b]$  and  $\sigma \in L_p(a, b)$  ( $1 \leq p \leq \infty$ ), the (left) Riemann–Liouville fractional integral operators satisfy the following relation:

$$(I_{a+}^\alpha I_{a+}^\beta \sigma)(x) = (I_{a+}^{\alpha+\beta} \sigma)(x)$$

for almost every point  $x \in [a, b]$ . If  $\alpha + \beta > 1$ , then the above relation holds at any point of  $[a, b]$ .

**Definition 2.2** ([27]). The (left) Riemann–Liouville fractional derivative  $D_{a+}^\omega \sigma$  of order  $\omega \in (m-1, m]$ ,  $m \in \mathbb{N}$ , is defined as

$$D_{a+}^\omega \sigma(t) = \frac{1}{\Gamma(m-\omega)} \frac{d^m}{dt^m} \int_a^t (t-s)^{m-1-\omega} \sigma(s) ds,$$

where  $\sigma, \sigma^{(m)} \in L^1[a, b]$  for  $-\infty \leq a < t < b \leq +\infty$ .

**Definition 2.3** ([27]). The (left) Caputo fractional derivative  ${}^C D_{a+}^\omega \sigma$  of order  $\omega \in (m-1, m]$ ,  $m \in \mathbb{N}$ , in terms of Riemann–Liouville fractional derivative operator  $D_{a+}^\omega$ , is defined as

$${}^C D_{a+}^\omega \sigma(t) = D_{a+}^\omega \left[ \sigma(t) - \sigma(a) - \sigma'(a) \frac{(t-a)}{1!} - \dots - \sigma^{(m-1)}(a) \frac{(t-a)^{m-1}}{(m-1)!} \right].$$

**Remark 2.2.** The (left) Caputo fractional derivative of order  $\omega \in (m-1, m]$ ,  $m \in \mathbb{N}$ , for a continuous function  $\sigma : [a, b] \rightarrow \mathbb{R}$  such that  $\sigma \in C^m[a, b]$ , existing almost everywhere on  $[a, b]$ , is defined by

$${}^C D_{a+}^\omega \sigma(t) = \frac{1}{\Gamma(m-\omega)} \int_a^t (t-s)^{m-\omega-1} \sigma^{(m)}(s) ds.$$

In the following lemma, we solve the linear variant of problem (1.1), (1.2). This lemma plays a key role in converting the given problem into a fixed point problem.

**Lemma 2.1.** For  $F, G \in C(\mathcal{J}, \mathbb{R})$  and  $\Delta \neq 0$ , the unique solution of the linear system

$$\begin{cases} {}^C D^{q_1} \varphi(t) = F(t), & t \in \mathcal{J}, \\ {}^C D^{q_2} \psi(t) = G(t), & t \in \mathcal{J}, \\ \varphi(0) = 0, \quad \varphi'(T) = 0, \quad \varphi(T) = \int_0^\xi [p_1 \psi(s) + T p_2 \psi'(s)] ds, \quad p_1, p_2 \in \mathbb{R}, \\ \psi(0) = 0, \quad \psi'(T) = 0, \quad \psi(T) = \int_0^\xi [r_1 \varphi(s) + T r_2 \varphi'(s)] ds, \quad r_1, r_2 \in \mathbb{R}, \end{cases} \quad (2.1)$$

is given by the following pair of integral equations:

$$\begin{aligned} \varphi(t) = & \int_0^t \frac{(t-v)^{q_1-1}}{\Gamma(q_1)} F(v) dv - \frac{1}{\Delta} \left\{ \alpha_1(t) \int_0^T \frac{(T-v)^{q_1-2}}{\Gamma(q_1-1)} F(v) dv \right. \\ & + \alpha_2(t) \left( p_1 \int_0^\xi \frac{(\xi-v)^{q_2}}{\Gamma(q_2+1)} G(v) dv + T p_2 \int_0^\xi \frac{(\xi-v)^{q_2-1}}{\Gamma(q_2)} G(v) dv ds - \int_0^T \frac{(T-v)^{q_1-1}}{\Gamma(q_1)} F(v) dv \right) \\ & + \alpha_3(t) \int_0^T \frac{(T-v)^{q_2-2}}{\Gamma(q_2-1)} G(v) dv \\ & \left. + \alpha_4(t) \left( r_1 \int_0^\xi \frac{(\xi-v)^{q_1}}{\Gamma(q_1+1)} F(v) dv + T r_2 \int_0^\xi \frac{(\xi-v)^{q_1-1}}{\Gamma(q_1)} F(v) dv - \int_0^T \frac{(T-v)^{q_2-1}}{\Gamma(q_2)} G(v) dv \right) \right\}, \quad (2.2) \end{aligned}$$

and

$$\begin{aligned} \psi(t) = & \int_0^t \frac{(t-v)^{q_2-1}}{\Gamma(q_2)} G(v) dv - \frac{1}{\Delta} \left\{ \beta_1(t) \int_0^T \frac{(T-v)^{q_1-2}}{\Gamma(q_1-1)} F(v) dv \right. \\ & + \beta_2(t) \left( p_1 \int_0^\xi \frac{(\xi-v)^{q_2}}{\Gamma(q_2+1)} G(v) dv + T p_2 \int_0^\xi \frac{(\xi-v)^{q_2-1}}{\Gamma(q_2)} G(v) dv - \int_0^T \frac{(T-v)^{q_1-1}}{\Gamma(q_1)} F(v) dv \right) \\ & + \beta_3(t) \int_0^T \frac{(T-v)^{q_2-2}}{\Gamma(q_2-1)} G(v) dv \\ & \left. + \beta_4(t) \left( r_1 \int_0^\xi \frac{(\xi-v)^{q_1}}{\Gamma(q_1+1)} F(v) dv + T r_2 \int_0^\xi \frac{(\xi-v)^{q_1-1}}{\Gamma(q_1)} F(v) dv - \int_0^T \frac{(T-v)^{q_2-1}}{\Gamma(q_2)} G(v) dv \right) \right\}, \quad (2.3) \end{aligned}$$

where

$$\begin{aligned} \alpha_1(t) &= [2TA_1B_2 - T^4 - A_2B_2]t + [T^3 - 2TA_1B_1 + A_2B_1]t^2, \\ \alpha_2(t) &= -2T^3t + T^2t^2, \\ \alpha_3(t) &= [-2T^3A_1 + 2T^2A_2]t + [T^2A_1 - TA_2]t^2, \\ \alpha_4(t) &= [-4T^2A_1 + 2TA_2]t + [2TA_1 - A_2]t^2, \\ \beta_1(t) &= [2T^2B_2 - 2T^3B_1]t + [T^2B_1 - TB_2]t^2, \\ \beta_2(t) &= [2TB_2 - 4T^2B_1]t + [2TB_1 - B_2]t^2, \\ \beta_3(t) &= [-T^4 - A_2B_2 + 2TA_2B_1]t + [T^3 + A_1B_2 - 2TA_1B_1]t^2, \\ \beta_4(t) &= -2T^3t + T^2t^2, \end{aligned} \quad (2.4)$$

and

$$\Delta = T^4 + 2TA_1B_2 - A_2B_2 - 4T^2A_1B_1 + 2TA_2B_1,$$

with

$$A_1 = p_1 \frac{\xi^2}{2} + Tp_2\xi, \quad A_2 = p_1 \frac{\xi^3}{3} + Tp_2\xi^2, \quad B_1 = r_1 \frac{\xi^2}{2} + Tr_2\xi, \quad B_2 = r_1 \frac{\xi^3}{3} + Tr_2\xi^2. \quad (2.5)$$

*Proof.* For arbitrary constants  $c_0, c_1, c_2, c_3, c_4, c_5 \in \mathbb{R}$ , it is well-known (see [27]) that the solution of fractional differential equations in (2.1) can be written as

$$\varphi(t) = \int_0^t \frac{(t-v)^{q_1-1}}{\Gamma(q_1)} F(v) dv - c_0 - c_1t - c_2t^2, \quad (2.6)$$

$$\psi(t) = \int_0^t \frac{(t-v)^{q_2-1}}{\Gamma(q_2)} G(v) dv - c_3 - c_4t - c_5t^2. \quad (2.7)$$

Using (??), (??) in the boundary conditions of problem (2.1) yields  $c_0 = 0$ ,  $c_3 = 0$  and a system of algebraic equations in  $c_1, c_2, c_4$  and  $c_5$  given by

$$\begin{aligned} c_1 + 2Tc_2 &= \int_0^T \frac{(T-v)^{q_1-2}}{\Gamma(q_1-1)} F(v) dv, \quad -Tc_1 - T^2c_2 + A_1c_4 + A_2c_5 = A_3, \\ c_4 + 2Tc_5 &= \int_0^T \frac{(T-v)^{q_2-2}}{\Gamma(q_2-1)} G(v) dv, \quad B_1c_1 + B_2c_2 - Tc_4 - T^2c_5 = B_3, \end{aligned} \quad (2.8)$$

where  $A_1, A_2, B_1, B_2$  are given in (2.5) and

$$\begin{aligned} A_3 &= p_1 \int_0^\xi \frac{(\xi-v)^{q_2}}{\Gamma(q_2+1)} G(v) dv + Tp_2 \int_0^\xi \frac{(\xi-v)^{q_2-1}}{\Gamma(q_2)} G(v) dv - \int_0^T \frac{(T-v)^{q_1-1}}{\Gamma(q_1)} F(v) dv, \\ B_3 &= r_1 \int_0^\xi \frac{(\xi-v)^{q_1}}{\Gamma(q_1+1)} F(v) dv + Tr_2 \int_0^\xi \frac{(\xi-v)^{q_1-1}}{\Gamma(q_1)} F(v) dv ds - \int_0^T \frac{(T-v)^{q_2-1}}{\Gamma(q_2)} G(v) dv. \end{aligned}$$

Solving system (??) for  $c_1, c_2, c_4$  and  $c_5$ , we obtain

$$\begin{aligned} c_1 &= \frac{1}{\Delta} \left\{ [2TA_1B_2 - T^4 - A_2B_2] \int_0^T \frac{(T-v)^{q_1-2}}{\Gamma(q_1-1)} F(v) dv \right. \\ &\quad - 2T^3 \left( p_1 \int_0^\xi \frac{(\xi-v)^{q_2}}{\Gamma(q_2+1)} G(v) dv + Tp_2 \int_0^\xi \frac{(\xi-v)^{q_2-1}}{\Gamma(q_2)} G(v) dv - \int_0^T \frac{(T-v)^{q_1-1}}{\Gamma(q_1)} F(v) dv \right) \\ &\quad + [2T^2A_2 - 2T^3A_1] \int_0^T \frac{(T-v)^{q_2-2}}{\Gamma(q_2-1)} G(v) dv + [2TA_2 - 4T^2A_1] \left( r_1 \int_0^\xi \frac{(\xi-v)^{q_1}}{\Gamma(q_1+1)} F(v) dv \right. \\ &\quad \left. \left. + Tr_2 \int_0^\xi \frac{(\xi-v)^{q_1-1}}{\Gamma(q_1)} F(v) dv ds - \int_0^T \frac{(T-v)^{q_2-1}}{\Gamma(q_2)} G(v) dv \right) \right\}, \\ c_2 &= \frac{1}{\Delta} \left\{ [T^3 - 2TA_1B_1 + A_2B_1] \int_0^T \frac{(T-v)^{q_1-2}}{\Gamma(q_1-1)} F(v) dv \right. \end{aligned}$$

$$\begin{aligned}
& + T^2 \left( p_1 \int_0^\xi \frac{(\xi - v)^{q_2}}{\Gamma(q_2 + 1)} G(v) dv + T p_2 \int_0^\xi \frac{(\xi - v)^{q_2-1}}{\Gamma(q_2)} G(v) dv - \int_0^T \frac{(T - v)^{q_1-1}}{\Gamma(q_1)} F(v) dv \right) \\
& + [T^2 A_1 - T A_2] \int_0^T \frac{(T - v)^{q_2-2}}{\Gamma(q_2 - 1)} G(v) dv + [2T A_1 - A_2] \left( r_1 \int_0^\xi \frac{(\xi - v)^{q_1}}{\Gamma(q_1 + 1)} F(v) dv \right. \\
& \quad \left. + T r_2 \int_0^\xi \frac{(\xi - v)^{q_1-1}}{\Gamma(q_1)} F(v) dv ds - \int_0^T \frac{(T - v)^{q_2-1}}{\Gamma(q_2)} G(v) dv \right) \Bigg\}, \\
c_4 = & \frac{1}{\Delta} \left\{ [2T^2 B_2 - 2T^3 B_1] \int_0^T \frac{(T - v)^{q_1-2}}{\Gamma(q_1 - 1)} F(v) dv \right. \\
& + [2T B_2 - 4T^2 B_1] \left( p_1 \int_0^\xi \frac{(\xi - v)^{q_2}}{\Gamma(q_2 + 1)} G(v) dv + T p_2 \int_0^\xi \frac{(\xi - v)^{q_2-1}}{\Gamma(q_2)} G(v) dv \right. \\
& \quad \left. - \int_0^T \frac{(T - v)^{q_1-1}}{\Gamma(q_1)} F(v) dv \right) + [2T A_2 B_1 - T^4 - A_2 B_2] \int_0^T \frac{(T - v)^{q_2-2}}{\Gamma(q_2 - 1)} G(v) dv \\
& \quad \left. - 2T^3 \left( r_1 \int_0^\xi \frac{(\xi - v)^{q_1}}{\Gamma(q_1 + 1)} F(v) dv + T r_2 \int_0^\xi \frac{(\xi - v)^{q_1-1}}{\Gamma(q_1)} F(v) dv ds - \int_0^T \frac{(T - v)^{q_2-1}}{\Gamma(q_2)} G(v) dv \right) \right\}, \\
c_5 = & \frac{1}{\Delta} \left\{ [T^2 B_1 - T B_2] \int_0^T \frac{(T - v)^{q_1-2}}{\Gamma(q_1 - 1)} F(v) dv \right. \\
& + [2T B_1 - B_2] \left( p_1 \int_0^\xi \frac{(\xi - v)^{q_2}}{\Gamma(q_2 + 1)} G(v) dv + T p_2 \int_0^\xi \frac{(\xi - v)^{q_2-1}}{\Gamma(q_2)} G(v) dv \right. \\
& \quad \left. - \int_0^T \frac{(T - v)^{q_1-1}}{\Gamma(q_1)} F(v) dv \right) + [T^3 + A_1 B_2 - 2T A_1 B_1] \int_0^T \frac{(T - v)^{q_2-2}}{\Gamma(q_2 - 1)} G(v) dv \\
& \quad \left. + T^2 \left( r_1 \int_0^\xi \frac{(\xi - v)^{q_1}}{\Gamma(q_1 + 1)} F(v) dv + T r_2 \int_0^\xi \frac{(\xi - v)^{q_1-1}}{\Gamma(q_1)} F(v) dv ds - \int_0^T \frac{(T - v)^{q_2-1}}{\Gamma(q_2)} G(v) dv \right) \right\},
\end{aligned}$$

Inserting the above values of  $c_i$ ,  $i = 1, 2, 4, 5$ , in (??), (??) together with (2.4), we get solution (2.2), (2.3). By direct computation, one can obtain the converse of the lemma.  $\square$

### 3 Main results

Let  $\Theta$  denote the Banach space of all continuous functions from  $\mathcal{J}$  to  $\mathbb{R}$  equipped with the supremum norm:  $\|\vartheta\| = \sup_{t \in \mathcal{J}} |\vartheta(t)|$ ,  $\vartheta \in \Theta$ . Then the product space  $\Theta \times \Theta$  is also a Banach space endowed with the norm

$$\|(\vartheta_1, \vartheta_2)\| = \|\vartheta_1\| + \|\vartheta_2\|, \quad (\vartheta_1, \vartheta_2) \in \Theta \times \Theta.$$

Associated with the nonlinear problem (1.1), (1.2), we define an operator  $\mathcal{H} : \Theta \times \Theta \rightarrow \Theta \times \Theta$  by

$$\mathcal{H}(\varphi, \psi)(t) = \begin{pmatrix} \mathcal{H}_1(\varphi, \psi)(t) \\ \mathcal{H}_2(\varphi, \psi)(t) \end{pmatrix}, \quad (3.1)$$

where

$$\begin{aligned}
\mathcal{H}_1(\varphi, \psi)(t) = & \int_0^t \frac{(t-v)^{q_1-1}}{\Gamma(q_1)} \rho_1(v, \varphi(v), \psi(v)) dv - \frac{1}{\Delta} \left\{ \alpha_1(t) \int_0^T \frac{(T-v)^{q_1-2}}{\Gamma(q_1-1)} \rho_1(v, \varphi(v), \psi(v)) dv \right. \\
& + \alpha_2(t) \left( p_1 \int_0^\xi \frac{(\xi-v)^{q_2}}{\Gamma(q_2+1)} \rho_2(v, \varphi(v), \psi(v)) dv + Tp_2 \int_0^\xi \frac{(\xi-v)^{q_2-1}}{\Gamma(q_2)} \rho_2(v, \varphi(v), \psi(v)) dv \right. \\
& \left. - \int_0^T \frac{(T-v)^{q_1-1}}{\Gamma(q_1)} \rho_1(v, \varphi(v), \psi(v)) dv \right) + \alpha_3(t) \int_0^T \frac{(T-v)^{q_2-2}}{\Gamma(q_2-1)} \rho_2(v, \varphi(v), \psi(v)) dv \\
& + \alpha_4(t) \left( r_1 \int_0^\xi \frac{(\xi-v)^{q_1}}{\Gamma(q_1+1)} \rho_1(v, \varphi(v), \psi(v)) dv + Tr_2 \int_0^\xi \frac{(\xi-v)^{q_1-1}}{\Gamma(q_1)} \rho_1(v, \varphi(v), \psi(v)) dv \right. \\
& \left. \left. - \int_0^T \frac{(T-v)^{q_2-1}}{\Gamma(q_2)} \rho_2(v, \varphi(v), \psi(v)) dv \right) \right\},
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{H}_2(\varphi, \psi)(t) = & \int_0^t \frac{(t-v)^{q_2-1}}{\Gamma(q_2)} \rho_2(v, \varphi(v), \psi(v)) dv - \frac{1}{\Delta} \left\{ \beta_1(t) \int_0^T \frac{(T-v)^{q_1-2}}{\Gamma(q_1-1)} \rho_1(v, \varphi(v), \psi(v)) dv \right. \\
& + \beta_2(t) \left( p_1 \int_0^\xi \frac{(\xi-v)^{q_2}}{\Gamma(q_2+1)} \rho_2(v, \varphi(v), \psi(v)) dv + Tp_2 \int_0^\xi \frac{(\xi-v)^{q_2-1}}{\Gamma(q_2)} \rho_2(v, \varphi(v), \psi(v)) dv \right. \\
& \left. - \int_0^T \frac{(T-v)^{q_1-1}}{\Gamma(q_1)} \rho_1(v, \varphi(v), \psi(v)) dv \right) + \beta_3(t) \int_0^T \frac{(T-v)^{q_2-2}}{\Gamma(q_2-1)} \rho_2(v, \varphi(v), \psi(v)) dv \\
& + \beta_4(t) \left( r_1 \int_0^\xi \frac{(\xi-v)^{q_1}}{\Gamma(q_1+1)} \rho_1(v, \varphi(v), \psi(v)) dv + Tr_2 \int_0^\xi \frac{(\xi-v)^{q_1-1}}{\Gamma(q_1)} \rho_1(v, \varphi(v), \psi(v)) dv \right. \\
& \left. \left. - \int_0^T \frac{(T-v)^{q_2-1}}{\Gamma(q_2)} \rho_2(v, \varphi(v), \psi(v)) dv \right) \right\}.
\end{aligned}$$

Here, it is imperative to mention that problem (1.1), (1.2) is equivalent to the fixed point problem:  $\mathcal{H}(\varphi, \psi) = (\varphi, \psi)$ , where the operator  $\mathcal{H} : \Theta \times \Theta \rightarrow \Theta \times \Theta$  is defined by (??). This means that a fixed point of the operator  $\mathcal{H}$  will be a solution to problem (1.1), (1.2).

To analyze problem (1.1), (1.2), we need the following hypotheses:

(H<sub>1</sub>) For  $\rho_1, \rho_2 \in C(\mathcal{J} \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$ , there exist real constants  $m_i, n_i \geq 0, i = 1, 2$ , and  $m_0, n_0 > 0$  such that

$$|\rho_1(t, \varphi, \psi)| \leq m_0 + m_1|\varphi| + m_2|\psi|, \quad |\rho_2(t, \varphi, \psi)| \leq n_0 + n_1|\varphi| + n_2|\psi|, \quad \forall \varphi, \psi \in \mathbb{R}.$$

(H<sub>2</sub>) There exist the constants  $\ell_i, \bar{\ell}_i, i = 1, 2$ , such that for all  $t \in \mathcal{J}, \varphi_i, \psi_i \in \mathbb{R}, i = 1, 2$ ,

$$\begin{aligned}
|\rho_1(t, \varphi_1, \psi_1) - \rho_1(t, \varphi_2, \psi_2)| & \leq \ell_1|\varphi_1 - \varphi_2| + \ell_2|\psi_1 - \psi_2|, \\
|\rho_2(t, \varphi_1, \psi_1) - \rho_2(t, \varphi_2, \psi_2)| & \leq \bar{\ell}_1|\varphi_1 - \varphi_2| + \bar{\ell}_2|\psi_1 - \psi_2|.
\end{aligned}$$



For computational convenience, we set the notation:

$$\begin{aligned}
\mathcal{Q}_1 &= \max_{t \in [0, T]} \left\{ \frac{t^{q_1}}{\Gamma(q_1 + 1)} + \frac{1}{|\Delta|} \left[ |\alpha_1(t)| \frac{T^{q_1-1}}{\Gamma(q_1)} + |\alpha_2(t)| \frac{T^{q_1}}{\Gamma(q_1 + 1)} \right. \right. \\
&\quad \left. \left. + |\alpha_4(t)| \left( |r_1| \frac{\xi^{q_1+1}}{\Gamma(q_1 + 2)} + |r_2| \frac{T \xi^{q_1}}{\Gamma(q_1 + 1)} \right) \right] \right\}, \\
\mathcal{Q}_2 &= \max_{t \in [0, T]} \frac{1}{|\Delta|} \left\{ |\alpha_2(t)| \left( |p_1| \frac{\xi^{q_2+1}}{\Gamma(q_2 + 2)} + |p_2| \frac{T \xi^{q_2}}{\Gamma(q_2 + 1)} \right) \right. \\
&\quad \left. + |\alpha_3(t)| \frac{T^{q_2-1}}{\Gamma(q_2)} + |\alpha_4(t)| \frac{T^{q_2}}{\Gamma(q_2 + 1)} \right\}, \\
\mathcal{Q}_3 &= \max_{t \in [0, T]} \left\{ \frac{t^{q_2}}{\Gamma(q_2 + 1)} + \frac{1}{|\Delta|} \left[ |\beta_2(t)| \left( |p_1| \frac{\xi^{q_2+1}}{\Gamma(q_2 + 2)} + |p_2| \frac{T \xi^{q_2}}{\Gamma(q_2 + 1)} \right) \right. \right. \\
&\quad \left. \left. + |\beta_3(t)| \frac{T^{q_2-1}}{\Gamma(q_2)} + |\beta_4(t)| \frac{T^{q_2}}{\Gamma(q_2 + 1)} \right] \right\}, \\
\mathcal{Q}_4 &= \max_{t \in [0, T]} \frac{1}{|\Delta|} \left\{ |\beta_1(t)| \frac{T^{q_1-1}}{\Gamma(q_1)} + |\beta_2(t)| \frac{T^{q_1}}{\Gamma(q_1 + 1)} \right. \\
&\quad \left. + |\beta_4(t)| \left( |r_1| \frac{\xi^{q_1+1}}{\Gamma(q_1 + 2)} + |r_2| \frac{T \xi^{q_1}}{\Gamma(q_1 + 1)} \right) \right\}
\end{aligned} \tag{3.2}$$

and

$$\mathcal{Q}_0 = \min \left\{ 1 - [m_1(\mathcal{Q}_1 + \mathcal{Q}_4) + n_1(\mathcal{Q}_2 + \mathcal{Q}_3)], 1 - [m_2(\mathcal{Q}_1 + \mathcal{Q}_4) + n_2(\mathcal{Q}_2 + \mathcal{Q}_3)] \right\}. \tag{3.3}$$

Now, we proceed to present our main results. In our first result, we prove the existence of solutions for problem (1.1), (1.2) by using the Leray-Schauder nonlinear alternative [17].

**Theorem 3.1.** *Assume that the condition  $(H_1)$  is satisfied. Then there exists at least one solution to problem (1.1), (1.2) on  $\mathcal{J}$ , provided that*

$$m_1(\mathcal{Q}_1 + \mathcal{Q}_4) + n_1(\mathcal{Q}_2 + \mathcal{Q}_3) < 1, \quad m_2(\mathcal{Q}_1 + \mathcal{Q}_4) + n_2(\mathcal{Q}_2 + \mathcal{Q}_3) < 1,$$

where  $\mathcal{Q}_i$ ,  $i = 1, 2, 3, 4$  are given in (3.1).

*Proof.* We verify the hypotheses of the Leray-Schauder nonlinear alternative [17] in different steps. Let us first show that the operator  $\mathcal{H} : \Theta \times \Theta \rightarrow \Theta \times \Theta$  defined by (??) is completely continuous. Note that the operator  $\mathcal{H}$  is continuous in view of continuity of the functions  $\rho_1$  and  $\rho_2$ . If  $\Upsilon \subset \Theta \times \Theta$  is bounded, then there exist the positive constants  $L_1$  and  $L_2$  such that  $|\rho_1(t, \varphi, \psi)| \leq L_1$ ,  $|\rho_2(t, \varphi, \psi)| \leq L_2$ ,  $\forall (\varphi, \psi) \in \Upsilon$ . Then, for any  $(\varphi, \psi) \in \Upsilon$ , we obtain

$$\begin{aligned}
\|\mathcal{H}_1(\varphi, \psi)\| &\leq \max_{t \in \mathcal{J}} \left\{ \int_0^t \frac{(t-v)^{q_1-1}}{\Gamma(q_1)} |\rho_1(v, \varphi(v), \psi(v))| dv \right. \\
&+ \frac{1}{|\Delta|} \left[ |\alpha_1(t)| \int_0^T \frac{(T-v)^{q_1-2}}{\Gamma(q_1-1)} |\rho_1(v, \varphi(v), \psi(v))| dv + |\alpha_2(t)| \left( |p_1| \int_0^\xi \frac{(\xi-v)^{q_2}}{\Gamma(q_2+1)} |\rho_2(v, \varphi(v), \psi(v))| dv \right. \right. \\
&\quad \left. \left. + T |p_2| \int_0^\xi \frac{(\xi-v)^{q_2-1}}{\Gamma(q_2)} |\rho_2(v, \varphi(v), \psi(v))| dv + \int_0^T \frac{(T-v)^{q_1-1}}{\Gamma(q_1)} |\rho_1(v, \varphi(v), \psi(v))| dv \right) \right. \\
&\quad \left. + |\alpha_3(t)| \int_0^T \frac{(T-v)^{q_2-2}}{\Gamma(q_2-1)} |\rho_2(v, \varphi(v), \psi(v))| dv + |\alpha_4(t)| \left( |r_1| \int_0^\xi \frac{(\xi-v)^{q_1}}{\Gamma(q_1+1)} |\rho_1(v, \varphi(v), \psi(v))| dv \right. \right.
\end{aligned}$$

$$\begin{aligned}
& + T|r_2| \left[ \int_0^\xi \frac{(\xi-v)^{q_1-1}}{\Gamma(q_1)} |\rho_1(v, \varphi(v), \psi(v))| dv + \int_0^T \frac{(T-v)^{q_2-1}}{\Gamma(q_2)} |\rho_2(v, \varphi(v), \psi(v))| dv \right] \Bigg\} \\
& \leq L_1 \max_{t \in [0, T]} \left\{ \frac{t^{q_1}}{\Gamma(q_1+1)} + \frac{1}{|\Delta|} \left[ |\alpha_1(t)| \frac{T^{q_1-1}}{\Gamma(q_1)} \right. \right. \\
& \quad \left. \left. + |\alpha_2(t)| \frac{T^{q_1}}{\Gamma(q_1+1)} + |\alpha_4(t)| \left( |r_1| \frac{\xi^{q_1+1}}{\Gamma(q_1+2)} + |r_2| \frac{T\xi^{q_1}}{\Gamma(q_1+1)} \right) \right] \right\} \\
& + L_2 \max_{t \in [0, T]} \frac{1}{|\Delta|} \left\{ |\alpha_2(t)| \left( |p_1| \frac{\xi^{q_2+1}}{\Gamma(q_2+2)} + |p_2| \frac{T\xi^{q_2}}{\Gamma(q_2+1)} \right) + |\alpha_3(t)| \frac{T^{q_2-1}}{\Gamma(q_2)} + |\alpha_4(t)| \frac{T^{q_2}}{\Gamma(q_2+1)} \right\} \\
& \leq L_1 \mathcal{Q}_1 + L_2 \mathcal{Q}_2. \tag{3.4}
\end{aligned}$$

In a similar fashion, one can obtain that

$$\|\mathcal{H}_2(\varphi, \psi)\| \leq L_1 \mathcal{Q}_4 + L_2 \mathcal{Q}_3. \tag{3.5}$$

From (??) and (??), it follows that

$$\|\mathcal{H}(\varphi, \psi)\| = \|\mathcal{H}_1(\varphi, \psi)\| + \|\mathcal{H}_2(\varphi, \psi)\| \leq L_1(\mathcal{Q}_1 + \mathcal{Q}_4) + L_2(\mathcal{Q}_2 + \mathcal{Q}_3),$$

which shows that  $\mathcal{H}(\Upsilon)$  is uniformly bounded.

To show that  $\mathcal{H}(\Upsilon)$  is equicontinuous, let  $t_1, t_2 \in \mathcal{J}$  with  $t_1 < t_2$ . Then we have

$$\begin{aligned}
& |\mathcal{H}_1(\varphi, \psi)(t_2) - \mathcal{H}_1(\varphi, \psi)(t_1)| \\
& \leq \left| \int_0^{t_2} \frac{(t_2-v)^{q_1-1}}{\Gamma(q_1)} \rho_1(v, \varphi(v), \psi(v)) dv - \int_0^{t_1} \frac{(t_1-v)^{q_1-1}}{\Gamma(q_1)} \rho_1(v, \varphi(v), \psi(v)) dv \right| \\
& + \frac{|t_2-t_1|}{|\Delta|} \left\{ \left[ 2TA_1B_2 - T^4 - A_2B_2 \right] + [T^3 - 2TA_1B_1 + A_2B_1](t_2+t_1) \left| \int_0^T \frac{(T-v)^{q_1-2}}{\Gamma(q_1-1)} |\rho_1(v, \varphi(v), \psi(v))| dv \right. \right. \\
& \quad \left. \left. + |T^2(t_2+t_1) - 2T^3| \left( |p_1| \int_0^\xi \frac{(\xi-v)^{q_2}}{\Gamma(q_2+1)} |\rho_2(v, \varphi(v), \psi(v))| dv \right. \right. \right. \\
& \quad \left. \left. + T|p_2| \int_0^\xi \frac{(\xi-v)^{q_2-1}}{\Gamma(q_2)} |\rho_2(v, \varphi(v), \psi(v))| dv + \int_0^T \frac{(T-v)^{q_1-1}}{\Gamma(q_1)} |\rho_1(v, \varphi(v), \psi(v))| dv \right) \right. \\
& \quad \left. + \left[ 2T^2A_2 - 2T^3A_1 \right] + [T^2A_1 - TA_2](t_2+t_1) \left| \int_0^T \frac{(T-v)^{q_2-2}}{\Gamma(q_2-1)} |\rho_2(v, \varphi(v), \psi(v))| dv \right. \right. \\
& \quad \left. \left. + \left| 2TA_2 - 4T^2A_1 + [2TA_1 - A_2](t_2+t_1) \right| \left( |r_1| \int_0^\xi \frac{(\xi-v)^{q_1}}{\Gamma(q_1+1)} |\rho_1(v, \varphi(v), \psi(v))| dv \right. \right. \right. \\
& \quad \left. \left. + T|r_2| \int_0^\xi \frac{(\xi-v)^{q_1-1}}{\Gamma(q_1)} |\rho_1(v, \varphi(v), \psi(v))| dv + \int_0^T \frac{(T-v)^{q_2-1}}{\Gamma(q_2)} |\rho_2(v, \varphi(v), \psi(v))| dv \right) \right\} \\
& \leq \frac{L_1}{\Gamma(q_1+1)} \left[ |(t_2-t_1)^{q_1} + t_2^{q_1} - t_1^{q_1}| + |(t_2-t_1)^{q_1}| \right] \\
& + \frac{L_1|t_2-t_1|}{|\Delta|\Gamma(q_1+2)} \left\{ \left| 2TA_1B_2 - T^4 - A_2B_2 + [T^3 - 2TA_1B_1 + A_2B_1](t_2+t_1) \right| q_1(q_1+1)T^{q_1-1} \right. \\
& \quad \left. + |T^2(t_2+t_1) - 2T^3| (q_1+1)T^{q_1} + |2TA_2 - 4T^2A_1 + [2TA_1 - A_2](t_2+t_1)| (|r_1|\xi^{q_1+1} + |r_2|(q_1+1)T\xi^{q_1}) \right\}
\end{aligned}$$

$$\begin{aligned}
& + \frac{L_2|t_2 - t_1|}{|\Delta|\Gamma(q_2 + 2)} \left\{ |T^2(t_2 + t_1) - 2T^3|(|p_1|\xi^{q_2+1} + |p_2|(q_2 + 1)T\xi^{q_2}) \right. \\
& \quad + |2T^2A_2 - 2T^3A_1 + [T^2A_1 - TA_2](t_2 + t_1)|q_2(q_2 + 1)T^{q_2-1} \\
& \quad \left. + |2TA_2 - 4T^2A_1 + [2TA_1 - A_2](t_2 + t_1)|(q_2 + 1)T^{q_2} \right\} \longrightarrow 0 \\
& \quad \text{as } t_2 - t_1 \rightarrow 0 \text{ independently of } (\varphi, \psi) \in \Upsilon.
\end{aligned}$$

In a similar manner, one can obtain

$$\begin{aligned}
& |\mathcal{H}_2(\varphi, \psi)(t_2) - \mathcal{H}_2(\varphi, \psi)(t_1)| \\
& \leq \left| \int_0^{t_2} \frac{(t_2 - v)^{q_2-1}}{\Gamma(q_2)} \rho_2(v, \varphi(v), \psi(v)) dv - \int_0^{t_1} \frac{(t_1 - v)^{q_2-1}}{\Gamma(q_2)} \rho_2(v, \varphi(v), \psi(v)) dv \right| \\
& + \frac{|t_2 - t_1|}{|\Delta|} \left\{ |2T^2B_2 - 2T^3B_1 + [T^2B_1 - TB_2](t_2 + t_1)| \int_0^T \frac{(T - v)^{q_1-2}}{\Gamma(q_1 + 1)} |\rho_1(v, \varphi(v), \psi(v))| dv \right. \\
& \quad + |2TB_2 - 4T^2B_1 + [2TB_1 - B_2](t_2 + t_1)| \left( |p_1| \int_0^\xi \frac{(\xi - v)^{q_2}}{\Gamma(q_2 + 1)} |\rho_2(v, \varphi(v), \psi(v))| dv \right. \\
& \quad \left. + T|p_2| \int_0^\xi \frac{(\xi - v)^{q_2-1}}{\Gamma(q_2)} |\rho_2(v, \varphi(v), \psi(v))| dv + \int_0^T \frac{(T - v)^{q_1-1}}{\Gamma(q_1)} |\rho_1(v, \varphi(v), \psi(v))| dv \right) \\
& \quad + |2TA_2B_1 - T^4 - A_2B_2 + [T^3 + A_1B_2 - 2TA_1B_1](t_2 + t_1)| \int_0^T \frac{(T - v)^{q_2-2}}{\Gamma(q_2 - 1)} |\rho_2(v, \varphi(v), \psi(v))| dv \\
& \quad + |T^2(t_2 + t_1) - 2T^3| \left( |r_1| \int_0^\xi \frac{(\xi - v)^{q_1}}{\Gamma(q_1 + 1)} |\rho_1(v, \varphi(v), \psi(v))| dv \right. \\
& \quad \left. + T|r_2| \int_0^\xi \frac{(\xi - v)^{q_1-1}}{\Gamma(q_1)} |\rho_1(v, \varphi(v), \psi(v))| dv + \int_0^T \frac{(T - v)^{q_2-1}}{\Gamma(q_2)} |\rho_2(v, \varphi(v), \psi(v))| dv \right) \Bigg\} \\
& \leq \frac{L_2}{\Gamma(q_2 + 1)} \left[ |(t_2 - t_1)^{q_2} + t_2^{q_2} - t_1^{q_2}| + |(t_2 - t_1)^{q_2}| \right] \\
& \quad + \frac{L_1|t_2 - t_1|}{|\Delta|\Gamma(q_1 + 2)} \left\{ |2T^2B_2 - 2T^3B_1 + [T^2B_1 - TB_2](t_2 + t_1)|q_1(q_1 + 1)T^{q_1-1} \right. \\
& \quad + |2TB_2 - 4T^2B_1 + [2TB_1 - B_2](t_2 + t_1)|(q_1 + 1)T^{q_1} \\
& \quad \left. + |T^2(t_2 + t_1) - 2T^3|(|r_1|\xi^{q_1+1} + |r_2|(q_1 + 1)T\xi^{q_1}) \right\} \\
& + \frac{L_2|t_2 - t_1|}{|\Delta|\Gamma(q_2 + 2)} \left\{ |2TB_2 - 4T^2B_1 + [2TB_1 - B_2](t_2 + t_1)|(|p_1|\xi^{q_2+1} + |p_2|(q_2 + 1)T\xi^{q_2}) \right. \\
& \quad \times |2TA_2B_1 - T^4 - A_2B_2 + [T^3 + A_1B_2 - 2TA_1B_1](t_2 + t_1)|q_2(q_2 + 1)T^{q_2-1} \\
& \quad \left. + |T^2(t_2 + t_1) - 2T^3|(q_2 + 1)T^{q_2} \right\} \longrightarrow 0 \text{ as } t_2 - t_1 \rightarrow 0 \text{ independently of } (\varphi, \psi) \in \Upsilon.
\end{aligned}$$

Thus,  $\mathcal{H}_1(\Upsilon)$  and  $\mathcal{H}_2(\Upsilon)$  are equicontinuous and hence  $\mathcal{H}(\Upsilon)$  is equicontinuous. Therefore, we deduce by the Arzelà–Ascoli theorem that  $\mathcal{H}(\Upsilon)$  is completely continuous.

In the final step, we consider a set  $\Xi = \{(\varphi, \psi) \in \Theta \times \Theta : (\varphi, \psi) = \zeta \mathcal{H}(\varphi, \psi), 0 < \zeta < 1\}$  and show that it is bounded. Let  $(\varphi, \psi) \in \Xi$ . Then  $(\varphi, \psi) = \zeta \mathcal{H}(\varphi, \psi)$  implies that  $\varphi(t) = \lambda \mathcal{H}_1(\varphi, \psi)(t)$

and  $\psi(t) = \zeta \mathcal{H}_2(\varphi, \psi)(t)$  for  $t \in \mathcal{J}$ . Thus, by the assumption  $(H_1)$ , we obtain

$$\begin{aligned}
|\varphi(t)| \leq \|\mathcal{H}_1(\varphi, \psi)\| \leq \max_{t \in \mathcal{J}} & \left\{ \int_0^t \frac{(t-v)^{q_1-1}}{\Gamma(q_1)} [m_0 + m_1|\varphi| + m_2|\psi|] dv \right. \\
& + \frac{1}{|\Delta|} \left[ |\alpha_1(t)| \int_0^T \frac{(T-v)^{q_1-2}}{\Gamma(q_1-1)} [m_0 + m_1|\varphi| + m_2|\psi|] dv \right. \\
& \quad + |\alpha_2(t)| \left( |p_1| \int_0^\xi \frac{(\xi-v)^{q_2}}{\Gamma(q_2+1)} [n_0 + n_1|\varphi| + n_2|\psi|] dv \right. \\
& \quad + T|p_2| \int_0^\xi \frac{(\xi-v)^{q_2-1}}{\Gamma(q_2)} [n_0 + n_1|\varphi| + n_2|\psi|] dv + \int_0^T \frac{(T-v)^{q_1-1}}{\Gamma(q_1)} [m_0 + m_1|\varphi| + m_2|\psi|] dv \Big) \\
& \quad + |\alpha_3(t)| \int_0^T \frac{(T-v)^{q_2-2}}{\Gamma(q_2-1)} [n_0 + n_1|\varphi| + n_2|\psi|] dv \\
& \quad + |\alpha_4(t)| \left( |r_1| \int_0^\xi \frac{(\xi-v)^{q_1}}{\Gamma(q_1+1)} [m_0 + m_1|\varphi| + m_2|\psi|] dv \right. \\
& \quad \left. \left. + T|r_2| \int_0^\xi \frac{(\xi-v)^{q_1-1}}{\Gamma(q_1)} [m_0 + m_1|\varphi| + m_2|\psi|] dv + \int_0^T \frac{(T-v)^{q_2-1}}{\Gamma(q_2)} [n_0 + n_1|\varphi| + n_2|\psi|] dv \right) \right] \Big\},
\end{aligned}$$

which implies that

$$\|\varphi\| \leq [m_0 + m_1\|\varphi\| + m_2\|\psi\|] \mathcal{Q}_1 + [n_0 + n_1\|\varphi\| + n_2\|\psi\|] \mathcal{Q}_2. \quad (3.6)$$

In a similar manner, one can find that

$$\|\psi\| \leq [n_0 + n_1\|\varphi\| + n_2\|\psi\|] \mathcal{Q}_3 + [m_0 + m_1\|\varphi\| + m_2\|\psi\|] \mathcal{Q}_4. \quad (3.7)$$

From (3.3), (3.4), it follows that

$$\|\varphi\| + \|\psi\| \leq \frac{m_0(\mathcal{Q}_1 + \mathcal{Q}_4) + n_0(\mathcal{Q}_2 + \mathcal{Q}_3)}{\mathcal{Q}_0},$$

where  $\mathcal{Q}_0$  is given by (3.2). As a consequence, we have

$$\|(\varphi, \psi)\| \leq \frac{m_0(\mathcal{Q}_1 + \mathcal{Q}_4) + n_0(\mathcal{Q}_2 + \mathcal{Q}_3)}{\mathcal{Q}_0}.$$

Therefore, the set  $\Xi$  is bounded. Since the hypotheses of the Leray–Schauder nonlinear alternative [17] are satisfied, we deduce by its conclusion that there exists at least one fixed point for the operator  $\mathcal{H}$ . Hence, problem (1.1), (1.2) admits a solution on  $\mathcal{J}$ .  $\square$

In the following result, we prove the uniqueness of solutions for problem (1.1), (1.2) by applying the Banach fixed point theorem.

**Theorem 3.2.** *Let the condition  $(H_2)$  be satisfied. If*

$$(\ell_1 + \ell_2)(\mathcal{Q}_1 + \mathcal{Q}_4) + (\bar{\ell}_1 + \bar{\ell}_2)(\mathcal{Q}_2 + \mathcal{Q}_3) < 1, \quad (3.8)$$

where  $\mathcal{Q}_i$ ,  $i = 1, 2, 3, 4$ , are given in (3.1), then problem (1.1), (1.2) has a unique solution on  $\mathcal{J}$ .

*Proof.* Fixing  $\sup_{t \in \mathcal{J}} \rho_1(t, 0, 0) = N_1 < \infty$  and  $\sup_{t \in \mathcal{J}} \rho_2(t, 0, 0) = N_2 < \infty$ , it follows by  $(H_2)$  that

$$\begin{aligned} |\rho_1(v, \varphi(v), \psi(v))| &= |\rho_1(v, \varphi(v), \psi(v)) - \rho_1(t, 0, 0) + \rho_1(t, 0, 0)| \leq \ell_1 \|\varphi\| + \ell_2 \|\psi\| + N_1, \\ |\rho_2(v, \varphi(v), \psi(v))| &= |\rho_2(v, \varphi(v), \psi(v)) - \rho_2(t, 0, 0) + \rho_2(t, 0, 0)| \leq \bar{\ell}_1 \|\varphi\| + \bar{\ell}_2 \|\psi\| + N_2, \end{aligned} \quad (3.9)$$

Next, we consider a closed ball  $\mathcal{U}_\delta = \{(\varphi, \psi) \in \Theta \times \Theta : \|(\varphi, \psi)\| \leq \delta\}$  with

$$\delta \geq \frac{N_1(\mathcal{Q}_1 + \mathcal{Q}_4) + N_2(\mathcal{Q}_2 + \mathcal{Q}_3)}{1 - [(\ell_1 + \ell_2)(\mathcal{Q}_1 + \mathcal{Q}_4) + (\bar{\ell}_1 + \bar{\ell}_2)(\mathcal{Q}_2 + \mathcal{Q}_3)]},$$

and show that  $\mathcal{H}(\mathcal{U}_\delta) \subset \mathcal{U}_\delta$ . For  $(\varphi, \psi) \in \Theta \times \Theta$ , by using (3.6), we obtain

$$\begin{aligned} \|\mathcal{H}_1(\varphi, \psi)\| &\leq \max_{t \in \mathcal{J}} \left\{ \int_0^t \frac{(t-v)^{q_1-1}}{\Gamma(q_1)} [\ell_1 \|\varphi\| + \ell_2 \|\psi\| + N_1] dv \right. \\ &\quad + \frac{1}{|\Delta|} \left[ |\alpha_1(t)| \int_0^T \frac{(T-v)^{q_1-2}}{\Gamma(q_1-1)} [\ell_1 \|\varphi\| + \ell_2 \|\psi\| + N_1] dv \right. \\ &\quad + |\alpha_2(t)| \left( |p_1| \int_0^\xi \frac{(\xi-v)^{q_2}}{\Gamma(q_2+1)} [\bar{\ell}_1 \|\varphi\| + \bar{\ell}_2 \|\psi\| + N_2] dv + T |p_2| \int_0^\xi \frac{(\xi-v)^{q_2-1}}{\Gamma(q_2)} [\bar{\ell}_1 \|\varphi\| + \bar{\ell}_2 \|\psi\| + N_2] dv \right. \\ &\quad + \int_0^T \frac{(T-v)^{q_1-1}}{\Gamma(q_1)} [\ell_1 \|\varphi\| + \ell_2 \|\psi\| + N_1] dv \Big) + |\alpha_3(t)| \int_0^T \frac{(T-v)^{q_2-2}}{\Gamma(q_2-1)} [\bar{\ell}_1 \|\varphi\| + \bar{\ell}_2 \|\psi\| + N_2] dv \\ &\quad + |\alpha_4(t)| \left( |r_1| \int_0^\xi \frac{(\xi-v)^{q_1}}{\Gamma(q_1+1)} [\ell_1 \|\varphi\| + \ell_2 \|\psi\| + N_1] dv \right. \\ &\quad + T |r_2| \int_0^\xi \frac{(\xi-v)^{q_1-1}}{\Gamma(q_1)} [\ell_1 \|\varphi\| + \ell_2 \|\psi\| + N_1] dv + \int_0^T \frac{(T-v)^{q_2-1}}{\Gamma(q_2)} [\bar{\ell}_1 \|\varphi\| + \bar{\ell}_2 \|\psi\| + N_2] dv \Big) \Big] \Big\} \\ &\leq [(\ell_1 + \ell_2)\delta + N_1] \mathcal{Q}_1 + [(\bar{\ell}_1 + \bar{\ell}_2)\delta + N_2] \mathcal{Q}_2. \end{aligned}$$

Likewise, we have

$$\|\mathcal{H}_2(\varphi, \psi)\| \leq [(\bar{\ell}_1 + \bar{\ell}_2)\delta + N_2] \mathcal{Q}_3 + [(\ell_1 + \ell_2)\delta + N_1] \mathcal{Q}_4.$$

Therefore, we get

$$\begin{aligned} \|\mathcal{H}(\varphi, \psi)\| &= \|\mathcal{H}_1(\varphi, \psi)\| + \|\mathcal{H}_2(\varphi, \psi)\| \\ &\leq [(\ell_1 + \ell_2)(\mathcal{Q}_1 + \mathcal{Q}_4) + (\bar{\ell}_1 + \bar{\ell}_2)(\mathcal{Q}_2 + \mathcal{Q}_3)] \delta + (\mathcal{Q}_1 + \mathcal{Q}_4) N_1 + (\mathcal{Q}_2 + \mathcal{Q}_3) N_2 \leq \delta, \end{aligned}$$

which shows that  $\mathcal{H}(\varphi, \psi) \in \mathcal{U}_\delta$ . Hence  $\mathcal{H}(\mathcal{U}_\delta) \subset \mathcal{U}_\delta$ .

Now, we will establish that the operator  $\mathcal{H}$  is a contraction. Toward this end, let  $(\varphi_1, \psi_1), (\varphi_2, \psi_2) \in \Theta \times \Theta$ . Then, for any  $t \in \mathcal{J}$ , we obtain

$$\begin{aligned} \|\mathcal{H}_1(\varphi_2, \psi_2) - \mathcal{H}_1(\varphi_1, \psi_1)\| &\leq \max_{t \in \mathcal{J}} \left\{ \int_0^t \frac{(t-v)^{q_1-1}}{\Gamma(q_1)} |\rho_1(v, \varphi_2(v), \psi_2(v)) - \rho_1(v, \varphi_1(v), \psi_1(v))| dv \right. \\ &\quad + \frac{1}{|\Delta|} \left[ |\alpha_1(t)| \int_0^T \frac{(T-v)^{q_1-2}}{\Gamma(q_1-1)} |\rho_1(v, \varphi_2(v), \psi_2(v)) - \rho_1(v, \varphi_1(v), \psi_1(v))| dv \right. \end{aligned}$$

$$\begin{aligned}
& + |\alpha_2(t)| \left( |p_1| \int_0^\xi \frac{(\xi-v)^{q_2}}{\Gamma(q_2+1)} |\rho_2(v, \varphi_2(v), \psi_2(v)) - \rho_2(v, \varphi_1(v), \psi_1(v))| dv \right. \\
& + T|p_2| \int_0^\xi \frac{(\xi-v)^{q_2-1}}{\Gamma(q_2)} |\rho_2(v, \varphi_2(v), \psi_2(v)) - \rho_2(v, \varphi_1(v), \psi_1(v))| dv \\
& + \int_0^T \frac{(T-v)^{q_1-1}}{\Gamma(q_1)} |\rho_1(v, \varphi_2(v), \psi_2(v)) - \rho_1(v, \varphi_1(v), \psi_1(v))| dv \Big) \\
& + |\alpha_3(t)| \int_0^T \frac{(T-v)^{q_2-2}}{\Gamma(q_2-1)} |\rho_2(v, \varphi_2(v), \psi_2(v)) - \rho_2(v, \varphi_1(v), \psi_1(v))| dv \\
& + |\alpha_4(t)| \left( |r_1| \int_0^\xi \frac{(\xi-v)^{q_1}}{\Gamma(q_1+1)} |\rho_1(v, \varphi_2(v), \psi_2(v)) - \rho_1(v, \varphi_1(v), \psi_1(v))| dv \right. \\
& + T|r_2| \int_0^\xi \frac{(\xi-v)^{q_1-1}}{\Gamma(q_1)} |\rho_1(v, \varphi_2(v), \psi_2(v)) - \rho_1(v, \varphi_1(v), \psi_1(v))| dv \\
& + \left. \int_0^T \frac{(T-v)^{q_2-1}}{\Gamma(q_2)} |\rho_2(v, \varphi_2(v), \psi_2(v)) - \rho_2(v, \varphi_1(v), \psi_1(v))| dv \right) \Big] \Big\} \\
& \leq (\ell_1 \|\varphi_2 - \varphi_1\| + \ell_2 \|\psi_2 - \psi_1\|) \max_{t \in \mathcal{J}} \left\{ \int_0^t \frac{(t-v)^{q_1-1}}{\Gamma(q_1)} dv + \frac{1}{|\Delta|} \left[ |\alpha_1(t)| \int_0^T \frac{(T-v)^{q_1-2}}{\Gamma(q_1-1)} dv \right. \right. \\
& + |\alpha_2(t)| \left( |p_1| \int_0^\xi \frac{(\xi-v)^{q_2}}{\Gamma(q_2+1)} dv + T|p_2| \int_0^\xi \frac{(\xi-v)^{q_2-1}}{\Gamma(q_2)} dv + \int_0^T \frac{(T-v)^{q_1-1}}{\Gamma(q_1)} dv \right) \\
& + |\alpha_3(t)| \int_0^T \frac{(T-v)^{q_2-2}}{\Gamma(q_2-1)} dv \\
& + |\alpha_4(t)| \left( |r_1| \int_0^\xi \frac{(\xi-v)^{q_1}}{\Gamma(q_1+1)} dv + T|r_2| \int_0^\xi \frac{(\xi-v)^{q_1-1}}{\Gamma(q_1)} dv + \int_0^T \frac{(T-v)^{q_2-1}}{\Gamma(q_2)} dv \right) \Big] \Big\} \\
& \leq [(\ell_1 + \ell_2)\mathcal{Q}_1 + (\bar{\ell}_1 + \bar{\ell}_2)\mathcal{Q}_2] (\|\varphi_2 - \varphi_1\| + \|\psi_2 - \psi_1\|).
\end{aligned}$$

Similarly, we can get

$$\|\mathcal{H}_2(\varphi_2, \psi_2) - \mathcal{H}_2(\varphi_1, \psi_1)\| \leq [(\ell_1 + \ell_2)\mathcal{Q}_4 + (\bar{\ell}_1 + \bar{\ell}_2)\mathcal{Q}_3] (\|\varphi_2 - \varphi_1\| + \|\psi_2 - \psi_1\|).$$

From the last two inequalities, it follows that

$$\|\mathcal{H}(\varphi_2, \psi_2) - \mathcal{H}(\varphi_1, \psi_1)\| \leq [(\ell_1 + \ell_2)(\mathcal{Q}_1 + \mathcal{Q}_4) + (\bar{\ell}_1 + \bar{\ell}_2)(\mathcal{Q}_2 + \mathcal{Q}_3)] (\|\varphi_2 - \varphi_1\| + \|\psi_2 - \psi_1\|),$$

which, in view of condition (3.5), implies that the operator  $\mathcal{H}$  is a contraction. So, by the Banach fixed point theorem, there exists a unique fixed point for the operator  $\mathcal{H}$  which is indeed a unique solution to problem (1.1), (1.2) on  $\mathcal{J}$ .  $\square$

### 3.1 Example

Consider a fully coupled fractional boundary value problem:

$$\begin{cases} {}^C D^{2.03} \varphi(t) = \rho_1(t, \varphi(t), \psi(t)), & t \in [0, 2], \\ {}^C D^{2.51} \psi(t) = \rho_2(t, \varphi(t), \psi(t)), & t \in [0, 2] \\ \varphi(0) = 0, \quad \varphi'(2) = 0, \quad \varphi(2) = \int_0^{\frac{3}{2}} \left[ \frac{5}{2} \psi(s) + \frac{8}{5} \psi'(s) \right] ds, \\ \psi(0) = 0, \quad \psi'(2) = 0, \quad \psi(2) = \int_0^{\frac{3}{2}} \left[ \frac{3}{10} \varphi(s) + \frac{4}{5} \varphi'(s) \right] ds, \end{cases} \quad (3.10)$$

where  $T = 2$ ,  $q_1 = 2.03$ ,  $q_2 = 2.51$ ,  $p_1 = \frac{5}{2}$ ,  $p_2 = \frac{4}{5}$ ,  $r_1 = \frac{3}{10}$ ,  $r_2 = \frac{2}{5}$ . With the given data, it is found that  $\mathcal{Q}_1 \approx 1.448$ ,  $\mathcal{Q}_2 \approx 3.6000$ ,  $\mathcal{Q}_3 \approx 1.332$  and  $\mathcal{Q}_4 \approx 1.141$  ( $\mathcal{Q}_i$ ,  $i = 1, 2, 3, 4$ , are given in (3.1)).

(a) We illustrate Theorem 3.1 by choosing

$$\begin{aligned} \rho_1(t, \varphi(t), \psi(t)) &= \frac{1}{2(t^2 + 6)} \frac{\varphi(t)|\varphi(t)|}{(1 + |\varphi(t)|)} + \frac{1}{15} \psi(t) \cos \psi(t) + \frac{1}{9\sqrt{t^2 + 9}}, \\ \rho_2(t, \varphi(t), \psi(t)) &= \frac{1}{23} \frac{\varphi(t)|\psi(t)|}{(1 + |\psi(t)|)} + \frac{\psi(t) \cos \varphi(t)}{(7 + t^2)^2} + \frac{1}{3(t + 4)^2}. \end{aligned} \quad (3.11)$$

From (??), it is easy to find that  $m_0 = \frac{1}{27}$ ,  $m_1 = \frac{1}{12}$ ,  $m_2 = \frac{1}{15}$ ,  $n_0 = \frac{1}{48}$ ,  $n_1 = \frac{1}{23}$ ,  $n_2 = \frac{1}{49}$ . Moreover,

$$m_1(\mathcal{Q}_1 + \mathcal{Q}_4) + n_1(\mathcal{Q}_2 + \mathcal{Q}_3) \approx 0.430 < 1 \quad \text{and} \quad m_2(\mathcal{Q}_1 + \mathcal{Q}_4) + n_2(\mathcal{Q}_2 + \mathcal{Q}_3) \approx 0.273 < 1.$$

Thus the hypotheses of Theorem 3.1 are satisfied and, consequently, there exists at least one solution to problem (3.7) with  $\rho_1(t, \varphi(t), \psi(t))$  and  $\rho_2(t, \varphi(t), \psi(t))$  given by (??) on  $[0, 2]$ .

(b) For demonstrating the application of Theorem 3.2, we take

$$\begin{aligned} \rho_1(t, \varphi(t), \psi(t)) &= \frac{e^{-t^2}}{(t^2 + 40)} \tan^{-1} \varphi(t) + \frac{1}{(35 + t^3)} \frac{|\psi(t)|}{(1 + |\psi(t)|)} + \frac{\cos t}{8\sqrt{t^3 + 2}}, \\ \rho_2(t, \varphi(t), \psi(t)) &= \frac{1}{(t^4 + 36)} \frac{|\varphi(t)|}{(1 + |\varphi(t)|)} + \frac{1}{\sqrt{t^2 + 625}} \cos \psi(t) + \frac{e^t}{2(\cos^2 t + 7)}. \end{aligned} \quad (3.12)$$

It follows from (??) that  $\ell_1 = \frac{1}{40}$ ,  $\ell_2 = \frac{1}{35}$ ,  $\bar{\ell}_1 = \frac{1}{36}$ ,  $\bar{\ell}_2 = \frac{1}{25}$  and

$$[(\ell_1 + \ell_2)(\mathcal{Q}_1 + \mathcal{Q}_4) + (\bar{\ell}_1 + \bar{\ell}_2)(\mathcal{Q}_2 + \mathcal{Q}_3)] \approx 0.473 < 1.$$

Clearly, the hypothesis of Theorem 3.2 is satisfied and hence its conclusion implies that there exists a unique solution to problem (3.7) with  $\rho_1(t, \varphi(t), \psi(t))$  and  $\rho_2(t, \varphi(t), \psi(t))$  given in (??) on  $[0, 2]$ .

## 4 Conclusion

In this paper, we have obtained the existence and uniqueness results for a Caputo type nonlinear fractional differential system supplemented with a new class of integral boundary conditions. The standard fixed point theorems are the main tools of our study. As a special case, our results correspond to the integral boundary conditions of the form

$$\varphi(0) = 0, \quad \varphi'(T) = 0, \quad \varphi(T) = \int_0^{T^-} [p_1 \psi(s) + T p_2 \psi'(s)] ds = p_1 \int_0^{T^-} \psi(s) ds + T p_2 \psi(T),$$

$$\psi(0) = 0, \quad \psi'(T) = 0, \quad \psi(T) = \int_0^{T^-} [r_1 \varphi(s) + Tr_2 \varphi'(s)] ds = r_1 \int_0^{T^-} \varphi(s) ds + Tr_2 \varphi(T),$$

in the limit  $\xi \rightarrow T^-$ , which are indeed new. If we take  $p_1 = 0 = r_1$  in the results of this paper, then we obtain the new ones associated with the following boundary conditions:

$$\begin{aligned} \varphi(0) = 0, \quad \varphi'(T) = 0, \quad \varphi(T) &= Tp_2 \int_0^\xi \psi'(s) ds = Tp_2 \psi(\xi), \\ \psi(0) = 0, \quad \psi'(T) = 0, \quad \psi(T) &= Tr_2 \int_0^\xi \varphi'(s) ds = Tr_2 \varphi(\xi). \end{aligned}$$

Thus, our results are not only new in the given configuration but also give rise to some new results as special cases by fixing the parameters involved in the problem at hand.

## Acknowledgements

The authors thank the reviewer for his/her constructive remarks on their work that led to its improvement.

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(Received 14.08.2024; revised 28.09.2024; accepted 10.10.2024)

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