Memoirs on Differential Equations and Mathematical Physics

Volume ??, 2026, 1–14

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EXISTENCE AND STABILITY RESULTS FOR PARTIAL ψ-FRACTIONAL INTEGRAL EQUATIONS

Abstract. We investigate a class of partial fractional integral equations involving the ψ-Riemann–Liouville fractional integral operator in Banach spaces. Existence and uniqueness of solutions are established via Banach's contraction principle under suitable Lipschitz conditions, while Schaefer's fixed point theorem is used to obtain existence results in the non-contractive case. We also prove Ulam–Hyers–Rassias type stability of solutions. The results extend several known fractional integral equations, including Hadamard-type models as special cases, and are illustrated by examples.

2020 Mathematics Subject Classification. 26A33, 34A08, 34B15, 47H10.

Key words and phrases. Integral Equations, ψ-Riemann–Liouville partial integral, existence, uniqueness, fixed point, generalized Ulam–Hyers–Rassias stability.

1 Introduction

Fractional calculus has become an important tool for describing nonlocal and memory–dependent processes. In contrast to classical integer–order models, fractional operators allow a more accurate representation of hereditary effects and long–range interactions. Fractional-order differential equations have therefore attracted considerable attention due to their wide-ranging applications in modeling various scientific and engineering phenomena, including blood flow dynamics, electrical circuits, biology, chemistry, physics, control theory, wave propagation, and signal and image processing. For more details, we refer the reader to [7, 8, 11–13, 19, 22, 35, 42].

In recent years, a large number of contributions has been devoted to both ordinary and partial fractional differential and integral equations, particularly concerning qualitative properties of their solutions such as existence, uniqueness, and stability. Comprehensive treatments can be found in the monographs of Abbas et al. [3, 5], Kilbas et al. [26], and Miller and Ross [28], as well as in the papers [1, 2, 6, 23, 36, 37, 41] and the references therein.

One of the main approaches in this area relies on fractional integral and derivative operators. Several operators have been introduced, including the Riemann–Liouville, Caputo, Hadamard, Katugampola and others (see [26]). Among these, the operators defined with respect to another function have recently attracted increasing attention because they unify many classical operators as particular cases. In 2000, Hilfer [24] introduced a generalized fractional derivative of order $a \in \mathbb{R}$ that interpolates between the Riemann–Liouville and Caputo derivatives. Later, Kilbas [26] formalized fractional derivatives and integrals with respect to another function, now known as the ψ-Riemann–Liouville fractional operators.

Fixed point theory plays a crucial role in the qualitative analysis of nonlinear equations. In particular, fixed point methods provide an efficient framework for treating nonlocal problems arising in fractional calculus. Banach's contraction principle [10] guarantees the existence and uniqueness of solutions for contraction operators in complete metric spaces. Schauder [33] and Tychonoff [38] later extended these ideas to compact operators in Banach spaces. These techniques have been successfully applied to many classes of fractional differential and integral equations.

The concept of stability also plays a fundamental role in the analysis of functional and differential equations. In 1940, Ulam [39, 40] raised the problem of stability of functional equations. Hyers [25] provided a partial answer in 1941, and Rassias [29, 30] later generalized the result. This notion, now known as Hyers-Ulam stability, has been extensively investigated for fractional models due to their sensitivity to perturbations (see [16–18, 23, 27, 31, 32, 34, 36, 37]).

In [4], S. Abbas *et al.* investigated the the existence and uniqueness of solutions to the following class of Hadamard partial fractional integral equations:

$$u(x,y) = \mu(x,y) + \frac{1}{\Gamma(\beta_1)\Gamma(\beta_2)} \int_1^x \int_1^y \left(\log\frac{x}{s}\right)^{\beta_1-1} \left(\log\frac{y}{t}\right)^{\beta_2-1} \frac{f(s,t,u(s,t))}{st}\, ds\, dt \quad \text{for} \quad (x,y) \in \mathcal{T}.$$

where $\mathcal{T} := [1,b] \times [1,c]$, $b, c > 0$, $\beta_1, \beta_2 > 0$, $\mu : \mathcal{T} \to \mathbb{R}$, $f : \mathcal{T} \times \mathbb{R} \to \mathbb{R}$ are the given continuous functions.

However, the results concerning partial integral equations involving ψ-Riemann–Liouville operators remain scarce in the literature, especially in Banach spaces. This motivates the present study.

In this paper, we investigate a class of partial fractional integral equations involving the ψ-Riemann–Liouville operator

$$\begin{aligned} u(x,y) &= g(x,y) \\ &+ \frac{1}{\Gamma(\beta_1)\Gamma(\beta_2)} \int_a^x \int_a^y \psi'(s)\psi'(t)(\psi(x)-\psi(s))^{\beta_1-1}(\psi(y)-\psi(t))^{\beta_2-1} f(s,t,u(s,t))\, ds\, dt, \end{aligned} \tag{1.1}$$

where $\mathcal{T} := [a,b] \times [a,c]$, $a, b, c > 0$, $\beta_1, \beta_2 > 0$, $g : \mathcal{T} \to \mathbb{R}$ and $f : \mathcal{T} \times \mathbb{R} \to \mathbb{R}$ are the given continuous functions.

The main objective of this work is to establish rigorous analytical results concerning the solvability and stability of problem (1.1) in Banach spaces.

The contributions of this paper are summarized as follows:

- We introduce a class of partial fractional integral equations governed by the ψ-Riemann–Liouville fractional integral operator in a Banach space setting. The proposed model provides a unified framework that includes several classical fractional integral equations as particular cases corresponding to specific choices of the function ψ.
- By transforming problem (1.1) into an equivalent operator equation, we construct a suitable operator on a Banach space of continuous functions. Under appropriate Lipschitz-type conditions imposed on the nonlinear term f, we apply Banach's contraction principle to prove the existence and uniqueness of solutions.
- When the contraction assumption is not satisfied, we establish the existence of solutions by applying Schaefer's fixed point theorem. For this purpose, we prove that the associated operator is continuous and completely continuous and the set of its possible fixed points is bounded.
- We investigate Ulam–Hyers type stability of the obtained solutions. Explicit estimates are derived showing that small perturbations in the integral equation generate controlled deviations in the corresponding solution.
- The general formulation allows reconstruct several known fractional models. In particular, the choice of $\psi(t) = \log(t)$ reduces problem (1.1) to Hadamard-type fractional integral equations studied in [3, 4]. Hence, the obtained results extend and complement previously known results in the literature.

The paper is organized as follows. Section 2 presents the necessary definitions and auxiliary results related to the ψ-Riemann–Liouville fractional integral operator. Section 3 is devoted to the existence and uniqueness of solutions using Banach's contraction principle and Schaefer's fixed point theorem. We also establish the Ulam–Hyers–Rassias stability and its generalized form. Finally, illustrative examples are provided to demonstrate the applicability of the theoretical results.

2 Preliminaries

In this section, we introduce some notations, definitions, and preliminary results that will be used throughout the paper.

Let $\mathcal{T} := [a, b] \times [a, c]$ with $a, b, c > 0$. We denote by $C(\mathcal{T}, \mathbb{R})$ the Banach space of all continuous functions from $\mathcal{T}$ to $\mathbb{R}$, equipped with the norm

$$\|u\|_\infty = \sup_{(x,y)\in\mathcal{T}} |u(x, y)|.$$

Similarly, $C^n(\mathcal{T}, \mathbb{R})$ denotes the space of functions that are n-times continuously differentiable on $\mathcal{T}$.

Let $\psi \in C^1([a, \max\{b, c\}], \mathbb{R})$ be an increasing function such that $\psi'(t) \neq 0$ for all $t \in [a, \max\{b, c\}]$.

We also consider the Banach space $L^1(\mathcal{T}, \mathbb{R})$ of all Lebesgue integrable functions $u : \mathcal{T} \to \mathbb{R}$, endowed with the norm

$$\|u\|_{L^1} = \int_a^b \int_a^c |u(x, y)|\, dy\, dx.$$

Define $\mathcal{T}_1 := \{(x, y, s, t) :\ 0 \le s \le x \le b,\ 0 \le t \le y \le c\}$ and the differential operators $D_1 := \frac{\partial}{\partial x}$, $D_2 := \frac{\partial}{\partial y}$, $D_1D_2 := \frac{\partial^2}{\partial x \partial y}$.

Definition 2.1 ([20, 21])**.** Let $\alpha > 0$ and $x \in L^1([a, b], \mathbb{R})$. The left-sided ψ-Riemann–Liouville fractional integral of order α with respect to the point a is defined by

$$\left(I^{\alpha,\psi}_{a^+} x\right)(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \psi'(s)(\psi(t) - \psi(s))^{\alpha-1} x(s)\, ds,$$

where $\Gamma(\,\cdot\,)$ denotes Euler's gamma function.

Definition 2.2 ([9]). Let $\beta = (\beta_1, \beta_2)$ with $\beta_1, \beta_2 > 0$. The ψ-Riemann–Liouville partial fractional integral of $u \in L^1(\mathcal{T}, \mathbb{R})$ is defined by

$$(\mathcal{I}_a^{\beta,\psi} u)(x, y) = \frac{1}{\Gamma(\beta_1)\Gamma(\beta_2)} \int_a^x \int_a^y \psi'(s)\psi'(t)(\psi(x) - \psi(s))^{\beta_1 - 1}(\psi(y) - \psi(t))^{\beta_2 - 1} u(s, t)\, ds\, dt.$$

Lemma 2.1 ([9]). *Let $\alpha_1, \alpha_2 \in (-1, +\infty)$ and $\beta = (\beta_1, \beta_2) \in (0, \infty) \times (0, \infty)$. Then we have the following equality:*

$$\begin{aligned} &\mathcal{I}_a^{\beta,\psi}\big((\psi(x) - \psi(a))^{\alpha_1}(\psi(y) - \psi(a))^{\alpha_2}\big) \\ &\qquad = \frac{\Gamma(\alpha_1 + 1)\Gamma(\alpha_2 + 1)}{\Gamma(\beta_1 + \alpha_1 + 1)\Gamma(\beta_2 + \alpha_2 + 1)} (\psi(x) - \psi(a))^{\beta_1 + \alpha_1}(\psi(y) - \psi(a))^{\beta_2 + \alpha_2}. \end{aligned}$$

Lemma 2.2 ([3]). *Let $\Upsilon \in C(\mathcal{T}, \mathbb{R}_+), Q, D_1Q, D_2Q, D_1D_2Q \in C(\mathcal{T}_\infty, \mathbb{R}_+)$, and let $c_0 > 0$ be a constant. If*

$$\Upsilon(x, y) \le c_0 + \int_a^x \int_a^y Q(x, y, s, t)\Upsilon(s, t)\, ds\, dt$$

for $(x, y) \in \mathcal{T}$, then

$$\Upsilon(x, y) \le c_0 \exp\Big(\int_a^x \int_a^y R(s, t)\, ds\, dt\Big),$$

where

$$R(x, y) = Q(x, y, x, y) + \int_a^x D_1Q(x, y, s, y)\, ds + \int_a^y D_2Q(x, y, x, t)\, dt + \int_a^x \int_a^y D_1D_2Q(x, y, s, t)\, ds\, dt.$$

Theorem 2.1 (Banach's Fixed Point Theorem [20,21]). *Let X be a Banach space and $\mathcal{A} : X \to X$ a contraction, i.e., there exists $\Lambda \in [0, 1)$ such that $\|\mathcal{A}(x_1) - \mathcal{A}(x_2)\| \le \Lambda\|x_1 - x_2\|$ for all $x_1, x_2 \in X$. Then $\mathcal{A}$ has a unique fixed point.*

Theorem 2.2 (Schaefer's Fixed Point Theorem [11–13,15]). *Let X be a Banach space and $\mathcal{A} : X \to X$ be a completely continuous operator. If the set $G = \{u \in X : u = \lambda\mathcal{A}u$ for some $\lambda \in (0, 1)\}$ is bounded, then $\mathcal{A}$ has at least one fixed point.*

3 Main results

3.1 Existence and Uniqueness of solutions

Let us start by defining what we mean by a solution of the integral equation (1.1).

Definition 3.1. A function $u \in C(\mathcal{T}, \mathbb{R})$ is said to be a solution of (1.1) if u satisfies the integral equation in (1.1) on $\mathcal{T}$.

We now prove our first theorem on the existence and uniqueness of a solution of problem (1.1) that is based on Banach's contraction principle.

Theorem 3.1. *Let the following hypothesis hold:*

($\mathbb{H}1$) *For any $u, v \in C(\mathcal{T}, \mathbb{R})$ and $(x, y) \in \mathcal{T}$, there exists $\kappa > 0$ such that*

$$|f(x, y, u(x, y)) - f(x, y, v(x, y))| \le \kappa|u(x, y) - v(x, y)|. \tag{3.1}$$

If

$$L := \frac{\kappa(\psi(b)-\psi(a))^{\beta_1}(\psi(c)-\psi(a))^{\beta_2}}{\Gamma(\beta_1+1)\Gamma(\beta_2+1)} < 1, \tag{3.2}$$

then there exists a unique solution of equation (1.1) on $\mathcal{T}$.

Proof. Transform integral equation (1.1) into a fixed point equation. Consider the operator $N : C \to C$ defined by

$$\begin{aligned}(Nu)(x,y) &= g(x,y)\\ &+ \frac{1}{\Gamma(\beta_1)\Gamma(\beta_2)}\int_a^x\int_a^y \psi'(s)\psi'(t)(\psi(x)-\psi(s))^{\beta_1-1}(\psi(y)-\psi(t))^{\beta_2-1}f(s,t,u(s,t))\,ds\,dt.\end{aligned} \tag{3.3}$$

Let $u, vinC(\mathcal{T},\mathbb{R})$. Then for $(x,y)\in\mathcal{T}$ we have

$$\begin{aligned}|(Nu)(x,y)-(Nv)(x,y)| &\leq \frac{1}{\Gamma(\beta_1)\Gamma(\beta_2)}\\ &\times\int_a^x\int_a^y \psi'(s)\psi'(t)(\psi(x)-\psi(s))^{\beta_1-1}(\psi(y)-\psi(t))^{\beta_2-1}\big|f(s,t,u(s,t))-f(s,t,v(s,t))\big|\,ds\,dt.\end{aligned}$$

By hypothesis (3.1), and using Lemma 2.1 with $\alpha_1=\alpha_2=0$, we get

$$\begin{aligned}|(Nu)(x,y)-(Nv)(x,y)| &\leq \frac{1}{\Gamma(\beta_1)\Gamma(\beta_2)}\int_a^x\int_a^y \psi'(s)\psi'(t)(\psi(x)-\psi(s))^{\beta_1-1}(\psi(y)-\psi(t))^{\beta_2-1}\kappa|u-v|\\ &\leq \frac{\kappa}{\Gamma(\beta_1+1)\Gamma(\beta_2+1)}(\psi(b)-\psi(a))^{\beta_1}(\psi(c)-\psi(a))^{\beta_2}|u-v|.\end{aligned}$$

Consequently, $|N(u)-N(v)|\leq L|u-v|$. From (3.2), N is a contraction, so N has a unique fixed point by Banach's contraction principle. □

Our second result is based on Schaefer's fixed point theorem.

Theorem 3.2. *Consider the following hypothesis to be verified:*

($\mathbb{H}2$) *There exist functions h_1, $h_2 \in C(\mathcal{T},\mathbb{R}_+)$ such that*

$$|f(x,y,u(x,y))| \leq h_1(x,y)+h_2(x,y)|u(x,y)| \text{ for any } (x,y)\in\mathcal{T}.$$

Then equation (1.1) has at least one solution defined on $\mathcal{T}$.

Proof. We use Schaefer's fixed point theorem to prove in several steps that the operator N defined in (3.3) has a fixed point.

Step 1: N is continuous.

Let $\{u_n\}_{n\in\mathbb{N}}$ be a sequence in C that converges to $u\in C$. Let $\varrho>0$ be such that $\|u_n\|_\infty\leq\varrho$. Then

$$\begin{aligned}&\big|(Nu_n)(x,y)-(Nu)(x,y)\big|\\ &\leq \frac{1}{\Gamma(\beta_1)\Gamma(\beta_2)}\int_a^x\int_a^y \psi'(s)\psi'(t)(\psi(x)-\psi(s))^{\beta_1-1}(\psi(y)-\psi(t))^{\beta_2-1}\\ &\qquad\times\big|f(s,t,u_n(s,t))-f(s,t,u(s,t))\big|\,ds\,dt\\ &\leq \frac{1}{\Gamma(\beta_1)\Gamma(\beta_2)}\sup_{(s,t)\in\mathcal{T}}\big|f(s,t,u_n(s,t))-f(s,t,u(s,t))\big|\end{aligned}$$

$$\times \int_a^x \int_a^y \psi'(s)\psi'(t)(\psi(x)-\psi(s))^{\beta_1-1}(\psi(y)-\psi(t))^{\beta_2-1}\, ds\, dt$$
$$\leq \frac{\kappa}{\Gamma(\beta_1+1)\Gamma(\beta_2+1)}(\psi(b)-\psi(a))^{\beta_1}(\psi(c)-\psi(a))^{\beta_2} \times \big\| f(\cdot,\cdot,u_n(\cdot,\cdot)) - f(\cdot,\cdot,u(\cdot,\cdot))\big\|_\infty.$$

From Lebesgue's dominated convergence theorem and the continuity of function f, we have $|(Nu_n)(x,y)-(Nu)(x,y)| \to 0$ as $n\to\infty$.

Step 2: N maps bounded sets to bounded sets in C.

Indeed, it is enough to show that for any $\zeta>0$, there exists a positive constant $\varkappa$ such that, for each $u\in B_\zeta = \{u\in C:\ \|u\|_\infty \leq \zeta\}$, we have $\|N(u)\|_\infty \leq \varkappa$. Set

$$h_i^* = \sup_{(x,y)\in\mathcal{T}} h_i(x,y), \quad i=1,2.$$

From $(\mathbb{H}2)$, for each $(x,y)\in\mathcal{T}$, we have

$$|(Nu)(x,y)| \leq |g(x,y)| + \frac{1}{\Gamma(\beta_1)\Gamma(\beta_2)} \int_a^x \int_a^y \psi'(s)\psi'(t)(\psi(x)-\psi(s))^{\beta_1-1}(\psi(y)-\psi(t))^{\beta_2-1}\big(h_1(s,t)+h_2(s,t)\|u\|_C\big)\, ds\, dt$$
$$\leq \|g\|_\infty + \frac{1}{\Gamma(\beta_1+1)\Gamma(\beta_2+1)}(\psi(b)-\psi(a))^{\beta_1}(\psi(c)-\psi(a))^{\beta_2}(h_1^*+h_2^*\zeta) := \varkappa.$$

Hence, $\|N(u)\| \leq \varkappa$.

Step 3: N maps bounded sets to equicontinuous sets in C.

Let $(x_1,x_2),(y_1,y_2)\in[a,b]\times[a,c]$, $x_1<x_2$, $y_1<y_2$, B_ζ be a bounded set of C as in Step 2, and let $u\in B_\zeta$. Then

$$\big|(Nu)(x_2,y_2)-(Nu)(x_1,y_1)\big| \leq \big|g(x_2,y_2)-g(x_1,y_1)\big|$$
$$+ \frac{1}{\Gamma(\beta_1)\Gamma(\beta_2)} \int_a^{x_1}\int_a^{y_1} \psi'(s)\psi'(t)\Big[(\psi(x_2)-\psi(s))^{\beta_1-1}(\psi(y_2)-\psi(t))^{\beta_2-1} - (\psi(x_1)-\psi(s))^{\beta_1-1}(\psi(y_1)-\psi(t))^{\beta_2-1}\Big]\,|f(s,t,u(s,t))|\, ds\, dt$$
$$+ \frac{1}{\Gamma(\beta_1)\Gamma(\beta_2)} \int_a^{x_1}\int_{y_1}^{y_2} \psi'(s)\psi'(t)(\psi(x_2)-\psi(s))^{\beta_1-1}(\psi(y_2)-\psi(t))^{\beta_2-1}|f(s,t,u(s,t))|\, ds\, dt$$
$$+ \frac{1}{\Gamma(\beta_1)\Gamma(\beta_2)} \int_{x_1}^{x_2}\int_a^{y_1} \psi'(s)\psi'(t)(\psi(x_2)-\psi(s))^{\beta_1-1}(\psi(y_2)-\psi(t))^{\beta_2-1}|f(s,t,u(s,t))|\, ds\, dt$$
$$+ \frac{1}{\Gamma(\beta_1)\Gamma(\beta_2)} \int_{x_1}^{x_2}\int_{y_1}^{y_2} \psi'(s)\psi'(t)(\psi(x_2)-\psi(s))^{\beta_1-1}(\psi(y_2)-\psi(t))^{\beta_2-1}|f(s,t,u(s,t))|\, ds\, dt.$$

Thus,

$$\big|(Nu)(x_2,y_2)-(Nu)(x_1,y_1)\big| \leq \big|g(x_2,y_2)-g(x_1,y_1)\big| + \frac{1}{\Gamma(\beta_1)\Gamma(\beta_2)} \int_a^{x_1}\int_a^{y_1} \psi'(s)\psi'(t)\Big|(\psi(x_2)-\psi(s))^{\beta_1-1}(\psi(y_2)-\psi(t))^{\beta_2-1}$$

$$
\begin{aligned}
&\qquad - (\psi(x_1)-\psi(s))^{\beta_1-1}(\psi(y_1)-\psi(t))^{\beta_2-1}\Big|(h_1^*+h_2^*\zeta)\,ds\,dt\\
&+\frac{1}{\Gamma(\beta_1)\Gamma(\beta_2)}\int_a^{x_1}\int_{y_1}^{y_2}\psi'(s)\psi'(t)(\psi(x_2)-\psi(s))^{\beta_1-1}(\psi(y_2)-\psi(t))^{\beta_2-1}(h_1^*+h_2^*\zeta)\,ds\,dt\\
&+\frac{1}{\Gamma(\beta_1)\Gamma(\beta_2)}\int_{x_1}^{x_2}\int_a^{y_1}\psi'(s)\psi'(t)(\psi(x_2)-\psi(s))^{\beta_1-1}(\psi(y_2)-\psi(t))^{\beta_2-1}(h_1^*+h_2^*\zeta)\,ds\,dt\\
&+\frac{1}{\Gamma(\beta_1)\Gamma(\beta_2)}\int_{x_1}^{x_2}\int_{y_1}^{y_2}\psi'(s)\psi'(t)(\psi(x_2)-\psi(s))^{\beta_1-1}(\psi(y_2)-\psi(t))^{\beta_2-1}(h_1^*+h_2^*\zeta)\,ds\,dt.
\end{aligned}
$$

Similarly, the following can be obtained:

$$
\begin{aligned}
&\big|(Nu)(x_2,y_2)-(Nu)(x_1,y_1)\big| \le \big|g(x_2,y_2)-g(x_1,y_1)\big| + \frac{h_1^*+h_2^*\zeta}{\Gamma(\beta_1)\Gamma(\beta_2)}\\
&\times\int_a^{x_1}\int_a^{y_1}\psi'(s)\psi'(t)\Big|(\psi(x_2)-\psi(s))^{\beta_1-1}(\psi(y_2)-\psi(t))^{\beta_2-1}-(\psi(x_1)-\psi(s))^{\beta_1-1}(\psi(y_1)-\psi(t))^{\beta_2-1}\Big|\,ds\,dt\\
&\qquad+\frac{h_1^*+h_2^*\zeta}{\Gamma(\beta_1+1)\Gamma(\beta_2+1)}\Big[(\psi(y_2)-\psi(y_1))^{\beta_2}\big((\psi(x_2)-\psi(a))^{\beta_1}-(\psi(x_2)-\psi(x_1))^{\beta_1}\big)\\
&+(\psi(x_2)-\psi(x_1))^{\beta_1}\big((\psi(y_2)-\psi(a))^{\beta_2}-(\psi(y_2)-\psi(y_1))^{\beta_2}\big)+(\psi(x_2)-\psi(x_1))^{\beta_1}(\psi(y_2)-\psi(y_1))^{\beta_2}\Big].
\end{aligned}
$$

As $x_1 \to x_2$ and $y_1 \to y_2$, the right-hand side of the preceding inequality tends to zero.

As a consequence of Steps 1–3, together with the Ascoli–Arzelà theorem, we can conclude that N is continuous and completely continuous.

Step 4: N is an a priori bound.

Now, it remains to show that the set $\mathcal{W}=\{u\in C:\ u=\widetilde{\lambda}N(u) \text{ for some } 0<\widetilde{\lambda}<1\}$ is bounded. Let $u\in C$ be such that $u=\widetilde{\lambda}N(u)$ for some $0<\widetilde{\lambda}<1$. Thus, for each $(x,y)\in\mathcal{T}$,

$$
\begin{aligned}
u(x,y)&=\widetilde{\lambda}\,g(x,y)\\
&\quad+\frac{\widetilde{\lambda}}{\Gamma(\beta_1)\Gamma(\beta_2)}\int_a^{x}\int_a^{y}\psi'(s)\psi'(t)(\psi(x)-\psi(s))^{\beta_1-1}(\psi(y)-\psi(t))^{\beta_2-1}f(s,t,u(s,t))\,ds\,dt.
\end{aligned}
$$

This implies that for each $(x,y)\in\mathcal{T}$ we have

$$
\begin{aligned}
|u(x,y)| &\le |g(x,y)| + \frac{1}{\Gamma(\beta_1)\Gamma(\beta_2)}\\
&\qquad\times\int_a^{x}\int_a^{y}\psi'(s)\psi'(t)(\psi(x)-\psi(s))^{\beta_1-1}(\psi(y)-\psi(t))^{\beta_2-1}(h_1(s,t)+h_2(s,t)|u(x,t)|)\,ds\,dt\\
&\le \|g\|_\infty + \frac{h_1^*}{\Gamma(\beta_1+1)\Gamma(\beta_2+1)}(\psi(b)-\psi(a))^{\beta_1}(\psi(c)-\psi(a))^{\beta_2}\\
&\qquad+\frac{h_2^*}{\Gamma(\beta_1)\Gamma(\beta_2)}\int_a^{x}\int_a^{y}\psi'(s)\psi'(t)(\psi(x)-\psi(s))^{\beta_1-1}(\psi(y)-\psi(t))^{\beta_2-1}|u(x,t)|\,ds\,dt\\
&\le C+\int_a^{x}\int_a^{y}Q(x,y,s,t)|u(x,t)|\,ds\,dt,
\end{aligned}
$$

where
$$C := \|g\|_\infty + \frac{h_1^*}{\Gamma(\beta_1+1)\Gamma(\beta_2+1)} (\psi(b)-\psi(a))^{\beta_1}(\psi(c)-\psi(a))^{\beta_2}$$
and
$$Q(x,y,s,t) = \frac{h_2^*}{\Gamma(\beta_1)\Gamma(\beta_2)} \psi'(s)\psi'(t)(\psi(x)-\psi(s))^{\beta_1-1}(\psi(y)-\psi(t))^{\beta_2-1}.$$
From Lemma 2.2, we obtain
$$|u(x,y)| \le C \exp\left(\int_a^x\int_a^y R(s,t)\,ds\,dt\right),$$
where
$$\begin{aligned} R(x,y) &= Q(x,y,x,y) + \int_a^x D_1Q(x,y,s,y)\,ds + \int_a^y D_2Q(x,y,x,t)\,dt + \int_a^x\int_a^y D_1D_2Q(x,y,s,t)\,ds\,dt \\ &\le \frac{h_2^*}{\Gamma(\beta_1)\Gamma(\beta_2)} \psi'(x)\psi'(y)(\psi(x)-\psi(a))^{\beta_1-1}(\psi(y)-\psi(a))^{\beta_2-1}. \end{aligned}$$

Hence,
$$\begin{aligned} |u(x,y)| &\le C\exp\left(\int_a^x\int_a^y \frac{h_2^*}{\Gamma(\beta_1)\Gamma(\beta_2)} \psi'(s)\psi'(t)(\psi(s)-\psi(a))^{\beta_1-1}(\psi(t)-\psi(a))^{\beta_2-1}\,ds\,dt\right) \\ &\le C\exp\left(\frac{h_2^*}{\Gamma(\beta_1+1)\Gamma(\beta_2+1)} (\psi(b)-\psi(a))^{\beta_1}(\psi(c)-\psi(a))^{\beta_2}\right) := \Re. \end{aligned}$$

As a consequence of Schaefer's fixed point theorem, we deduce that N has a fixed point which is a solution of equation (1.1). □

3.2 Generalized Ulam-type stability

Now, we concern the generalized Ulam–Hyers–Rassias Stability of our problem (1.1). The following conditions will be used in the sequel:

($\mathfrak{H}1$) There exist the functions $l_1, l_2 \in C(\mathcal{T}, \mathbb{R}_+)$ such that for any $u \in C$ and $(x,y) \in \mathcal{T}$,
$$\big(1+|u(x,y)|\big)|f(x,y,u)| \le l_1(x,y) + l_2(x,y)|u(x,y)|.$$

($\mathfrak{H}2$) There exists $\varpi > 0$ such that for each $(x,y) \in \mathcal{T}$, we have
$$(\mathcal{I}_a^{\beta,\psi}\Theta)(x,y) \le \varpi\Theta(x,y).$$

Recall $N : C \to C$ as defined in (3.3). Let $\Theta : \mathcal{T} \to [0,\infty)$ be a continuous function. We consider the inequality
$$\big|u(x,y) - (Nu)(x,y)\big| \le \Theta(x,y), \quad (x,y) \in \mathcal{T}. \tag{3.4}$$

Definition 3.2 ([4, 34])**.** Equation (1.1) is Ulam–Hyers–Rassias stable with respect to Θ if there exists a real number $C_{N,\Theta} > 0$ such that for each $\varepsilon > 0$ and for each solution $u \in C(\mathcal{T},\mathbb{R})$ of (3.4) there exists a solution $v \in C(\mathcal{T},\mathbb{R})$ of equation (1.1) with
$$|u(x,y) - v(x,y)| \le \varepsilon C_{N,\Theta}\Theta(x,y), \quad (x,y) \in \mathcal{T}.$$

Definition 3.3 ([4, 34])**.** Equation (1.1) is generalized Ulam–Hyers–Rassias stable with respect to Θ if there exists a real number $C_{N,\Theta} > 0$ such that for each solution $u \in C(\mathcal{T},\mathbb{R})$ of (3.4), there exists a solution $v \in C(\mathcal{T},\mathbb{R})$ of equation (1.1) with
$$|u(x,y) - v(x,y)| \le C_{N,\Theta}\Theta(x,y), \quad (x,y) \in \mathcal{T}.$$

Remark 3.3. It is clear that Definition 3.2 implies Definition 3.3.

Theorem 3.4. *Assume* $(\mathfrak{H}1)$, $(\mathfrak{H}2)$ *and* $(\mathbb{H}2)$ *hold. Furthermore, suppose that there exist* $\widetilde{l}_i \in C(\mathcal{T}, \mathbb{R}_+)$, $i = 1, 2$, *such that for each* $(x, y) \in \mathcal{T}$, *we have*

$$l_i(x,y) \leq \widetilde{l}_i(x,y)\Theta(x,y). \tag{3.5}$$

Then integral equation (1.1) is generalized Ulam–Hyers–Rassias stable.

Proof. Let u be a solution of inequality (3.4). By Theorem 3.2, there exists w that is a solution of integral equation (1.1). Hence,

$$w(x,y) = g(x,y) + \frac{1}{\Gamma(\beta_1)\Gamma(\beta_2)} \times \int_a^x \int_a^y \psi'(s)\psi'(t)(\psi(x)-\psi(s))^{\beta_1-1}(\psi(y)-\psi(t))^{\beta_2-1} f(s,t,w(s,t))\, ds\, dt.$$

By inequality (3.4), for each $(x, y) \in \mathcal{T}$, we have

$$\left| u(x,y) - g(x,y) - \int_a^x \int_a^y \frac{\psi'(s)\psi'(t)(\psi(x)-\psi(s))^{\beta_1-1}(\psi(y)-\psi(t))^{\beta_2-1} f(s,t,u)}{\Gamma(\beta_1)\Gamma(\beta_2)}\, ds\, dt \right| \leq \Theta(x,y).$$

Set

$$\widetilde{\boldsymbol{l}}_{\boldsymbol{i}}^{\boldsymbol{*}} = \sup_{(x,y)\in\mathcal{T}} \widetilde{l}_i(x,y), \quad i = 1, 2.$$

For each $(x, y) \in \mathcal{T}$, we have

$$\begin{aligned}
|u(x,y) - w(x,y)| &\leq \Big| u(x,y) - g(x,y) \\
&\quad - \frac{1}{\Gamma(\beta_1)\Gamma(\beta_2)} \int_a^x \int_a^y \psi'(s)\psi'(t)(\psi(x)-\psi(s))^{\beta_1-1}(\psi(y)-\psi(t))^{\beta_2-1} f(s,t,u)\, ds\, dt \Big| \\
&\quad + \frac{1}{\Gamma(\beta_1)\Gamma(\beta_2)} \int_a^x \int_a^y \psi'(s)\psi'(t)(\psi(x)-\psi(s))^{\beta_1-1}(\psi(y)-\psi(t))^{\beta_2-1} \\
&\qquad \times \big| f(s,t,u) - f(s,t,w) \big|\, ds\, dt \\
&\leq \Theta(x,y) + \frac{1}{\Gamma(\beta_1)\Gamma(\beta_2)} \int_a^x \int_a^y \psi'(s)\psi'(t)(\psi(x)-\psi(s))^{\beta_1-1}(\psi(y)-\psi(t))^{\beta_2-1} \\
&\qquad \times \Big(2\widetilde{\boldsymbol{l}}_{\boldsymbol{1}}^{\boldsymbol{*}} + \frac{\widetilde{\boldsymbol{l}}_{\boldsymbol{2}}^{\boldsymbol{*}}|u(s,t)|}{1+|u(s,t)|} + \frac{\widetilde{\boldsymbol{l}}_{\boldsymbol{2}}^{\boldsymbol{*}}|w(s,t)|}{1+|w(s,t)|} \Big) \Theta(s,t)\, ds\, dt \\
&\leq \Theta(x,y) + 2(\widetilde{\boldsymbol{l}}_{\boldsymbol{1}}^{\boldsymbol{*}} + \widetilde{\boldsymbol{l}}_{\boldsymbol{2}}^{\boldsymbol{*}})(\mathcal{I}_a^{\beta,\psi}\Theta)(x,y) \\
&\leq \big(1 + 2(\widetilde{\boldsymbol{l}}_{\boldsymbol{1}}^{\boldsymbol{*}} + \widetilde{\boldsymbol{l}}_{\boldsymbol{2}}^{\boldsymbol{*}})\varpi\big)\Theta(x,y) \\
&:= C_{N,\Theta}\Theta(x,y).
\end{aligned}$$

Hence, integral equation (1.1) is generalized Ulam-Hyers-Rassias stable. □

4 Illustrative examples

In this section, we provide three concrete examples to illustrate the applicability of the main theoretical results obtained in this work, particularly focusing on the existence and stability of solutions to partial ψ-Riemann–Liouville fractional integral equations.

Example 4.1. As an application of our results, consider the following partial ψ-Riemann–Liouville integral equation:

$$u(x,y) = g(x,y) + \frac{1}{\Gamma(\beta_1)\Gamma(\beta_2)}$$

$$\times \int_0^x \int_0^y \psi'(s)\psi'(t)(\psi(x)-\psi(s))^{\beta_1-1}(\psi(y)-\psi(t))^{\beta_2-1} f(s,t,u(s,t))\, ds\, dt \ \text{ for } \ (x,y) \in [0,1]\times[0,1],$$

where $(\beta_1,\beta_2) = (0.5, 0.5)$, $\psi(t) = t$, $g(x,y) = x^2 + y^2$, and

$$f(x,y,u(x,y)) = xy^2 + \frac{u(x,y)}{e^{x+y+2}(1+|u(x,y)|)}\,.$$

For any $u, v \in C(\mathcal{T}, \mathbb{R})$ and $(x,y) \in [0,1]\times[0,1]$, we have

$$\big|f(x,y,u) - f(x,y,v)\big| \le e^{-2}|u(x,y) - v(x,y)|.$$

Thus, condition $(\mathbb{H}1)$ is satisfied with $\kappa = e^{-2}$. Consequently, we compute

$$L := \frac{\kappa(\psi(1)-\psi(0))^{\beta_1}(\psi(1)-\psi(0))^{\beta_2}}{\Gamma(\beta_1+1)\Gamma(\beta_2+1)} = \frac{4}{\pi e^2} < 1.$$

Hence, by Theorem 3.1, the integral equation (1.1) admits a unique solution defined on $[0,1]\times[0,1]$.

Moreover, for any $u \in C(\mathcal{T}, \mathbb{R})$ and $(x,y) \in [0,1]\times[0,1]$, we have

$$|f(x,y,u)| \le xy^2 + \frac{e^{-(x+y)}}{1+|u(x,y)|}\,|u(x,y)|.$$

Thus, condition $(\mathfrak{H}1)$ is satisfied with $l_1(x,y) = xy^2$ and $l_2(x,y) = e^{-(x+y)}$.

Similarly, condition $(\mathfrak{H}2)$ holds with $\Theta(x,y) = e^2$ and $\varpi = \frac{4}{\pi}$. In fact, for every $(x,y) \in [0,1]\times[0,1]$, we obtain

$$(\mathcal{I}_a^{\beta,\psi}\Theta)(x,y) \le \frac{4}{\pi}\, e^2 = \varpi\Theta(x,y).$$

Therefore, condition (3.5) is satisfied with $\widetilde{l}_1(x,y) = xy$ and $\widetilde{l}_2(x,y) = e^{x+y}$.

Consequently, Theorem 3.4 ensures that the integral equation (1.1) is generalized Ulam–Hyers–Rassias stable.

Example 4.2. Consider the following partial ψ-Riemann–Liouville fractional integral equation:

$$u(x,y) = g(x,y) + \frac{1}{\Gamma(\beta_1)\Gamma(\beta_2)}$$

$$\times \int_0^x \int_0^y \psi'(s)\psi'(t)(\psi(x)-\psi(s))^{\beta_1-1}(\psi(y)-\psi(t))^{\beta_2-1} f(s,t,u(s,t))\, ds\, dt \ \text{ for } \ (x,y) \in [0,1]\times[0,1],$$

with the following data: $(\beta_1,\beta_2) = (0.8, 0.6)$, $\psi(t) = \sqrt{t+1}\,$, $g(x,y) = \sin(xy)$, and the nonlinear function:

$$f(x,y,u(x,y)) = \cos(x+y) + \frac{xy\,u(x,y)}{1+|u(x,y)|}\,.$$

We observe that

$$|\cos(x+y)| \le 1 \ \text{ and } \ \left|\frac{xy\,u(x,y)}{1+|u(x,y)|}\right| \le xy\,|u(x,y)|.$$

For any $(x,y) \in [0,1]\times[0,1]$, we have $|f(x,y,u(x,y))| \le 1 + xy\,|u(x,y)|$. Thus, condition $(\mathbb{H}2)$ is satisfied with $h_1(x,y) = 1$ and $h_2(x,y) = xy$. Consequently, Theorem 3.2 implies that the integral equation (1.1) has at least one solution defined on $[0,1]\times[0,1]$.

Moreover, for any $u \in C(\mathcal{T}, \mathbb{R})$ and $(x, y) \in [0,1] \times [0,1]$, we have

$$|f(x, y, u)| \leq 1 + \frac{|u(x, y)|}{1 + |u(x, y)|}.$$

Thus, condition $(\mathfrak{H}1)$ is satisfied with $l_1(x, y) = l_2(x, y) = 1$.

Similarly, condition $(\mathfrak{H}2)$ holds with $\Theta(x, y) = e^3$ and $\varpi = 1$. In fact, for every $(x, y) \in [0,1] \times [0,1]$, we obtain

$$(\mathcal{I}_{a}^{\beta,\psi}\Theta)(x, y) \leq 0,48\big(\sqrt{x+1} - 1\big)^{0,8}\big(\sqrt{y+1} - 1\big)^{0,6} e^3 \leq \varpi\Theta(x, y).$$

Therefore, condition (3.5) is satisfied with $\widetilde{l}_1(x, y) = \widetilde{l}_2(x, y) = 1$.

Consequently, Theorem 3.4 ensures that the integral equation (1.1) is generalized Ulam-Hyers-Rassias stable.

Conclusion

In this paper, we studied a class of partial fractional integral equations involving the ψ-Riemann–Liouville fractional integral operator in the Banach space $C(\mathcal{T}, \mathbb{R})$. By reformulating the problem as an equivalent operator equation, we established the existence and uniqueness results under suitable Lipschitz conditions via Banach's contraction principle. In addition, by verifying the continuity, boundedness, and compactness of the associated operator, we applied Schaefer's fixed point theorem to derive the existence results in the non-contractive case.

Furthermore, we obtained the Ulam–Hyers–Rassias and generalized Ulam–Hyers–Rassias stability results, providing explicit bounds that characterize the dependence of solutions on perturbations. The proposed framework unifies and extends several existing fractional integral models in the literature; in particular, previously studied equations are recovered as special cases for suitable choice of the function ψ. Therefore, the present study contributes to the qualitative analysis of partial fractional integral equations with respect to another function.

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(Received 20.08.2025; revised 21.02.2026; accepted 27.02.2026)

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